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Triangle-Conic Porism

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Abstract. We investigate, for a given triangle, inscribed triangles whose sides are tangent to a given conic.

Consider a triangle $A_1B_1C_1$ inscribed in ABC, and a conic $\mathcal{C}$ inscribed in $A_1B_1C_1$. We ask whether there are other inscribed triangles in ABC and tritangent to $\mathcal{C}$. Restricting to circles, Ton Lecluse wrote about this problem in [6]. See also [5]. He suggested after use of dynamic geometry software that in general there is a second triangle tritangent to $\mathcal{C}$ and inscribed in ABC. In this paper we answer Lecluse's question.

Proposition 1. *Let $A'B'C'$ be a variable triangle of which B' and C' lie on CA and AB respectively. If the sidelines of triangle $A'B'C'$ are tangent to a conic $\mathcal{C}$, then the locus of A' is either a conic or a line.*

Proof. Let XYZ be the points on $\mathcal{C}$ and where $C'A'$, $A'B'$, and $B'C'$ respectively meet $\mathcal{C}$. ZX is the polar (with respect to $\mathcal{C}$) of B', which passes through a fixed point P_B, the pole of CA. Similarly XY passes through a fixed point P_C. The mappings $Y \mapsto X$ and $X \mapsto Z$ are thus involutions on $\mathcal{C}$. Hence $Y \mapsto Z$ is a projectivity. That means that the lines YZ form a pencil of lines or envelope a conic according as $Y \mapsto Z$ is an involution or not. Consequently the poles of these lines, the points A', run through a line ℓ_A or a conic $\mathcal{C}_A$. □

Two degenerate triangles $A'B'C'$, corresponding to the tangents from A, arise as limit cases. Hence, when $Y \mapsto Z$ is an involution, the points U_1 and U_2 of contact of tangents from A to $\mathcal{C}$ are its fixed points, and $\ell_A = U_1U_2$ is the polar of A.

The conics $\mathcal{C}$ and $\mathcal{C}_A$ are tangent to each other in U_1 and U_2. We see that $\mathcal{C}$ and $\mathcal{C}_A$ generate a pencil, of which the pair of common tangents, and the polar of A (as double line) are the degenerate elements. In view of this we may consider the line ℓ_A as a conic $\mathcal{C}_A$ degenerated into a "double" line. We do so in the rest of this paper.

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Proposition 1 shows us that if there is one inscribed triangle tritangent to $\mathcal{C}$, there will be in general another such triangle. This answers Lecluse's question for the general case. But it turns out that the other cases lead to interesting configurations as well.

The number of intersections of $\mathcal{C}_A$ with BC gives the number of inscribed triangles tritangent to $\mathcal{C}$. There may be infinitely many, if $\mathcal{C}_A$ degenerates and contains BC. This implies that $BC = \ell_A$. By symmetry it is necessary that ABC is self-polar with respect to $\mathcal{C}$. Of course this applies also when the above A' runs through ℓ_A in the plane of the triangle bounded by AB, CA and ℓ_A.

There are two possibilities for $\mathcal{C}_A$ and BC to intersect in one "double" point. One is that $\mathcal{C}_A$ is nondegenerate and tangent to BC. In this case, by reasons of continuity, the point of tangency belongs to one triangle $A'B'C'$, and similar conics $\mathcal{C}_B$ and $\mathcal{C}_C$ are tangent to the corresponding side as well. The points of tangency form the cevian triangle of the perspector of $\mathcal{C}$.

This can be seen by considering the point M where U_1U_2 meets BC. The polar of M with respect to $\mathcal{C}$ passes through the pole of U_1U_2, and through the intersections of the polars of B and C, hence the pole of BC. So the polar ℓ_M of M is the A-cevian of the perspector[1] of $\mathcal{C}$. The point where U_1U_2 and ℓ_M meet is the harmonic conjugate of M with respect to U_1 and U_2. This all applies to $\mathcal{C}_A$ as well. In case $\mathcal{C}_A$ is tangent to BC, the point of tangency is the pole of BC, and is thus the trace of the perspector of $\mathcal{C}$.

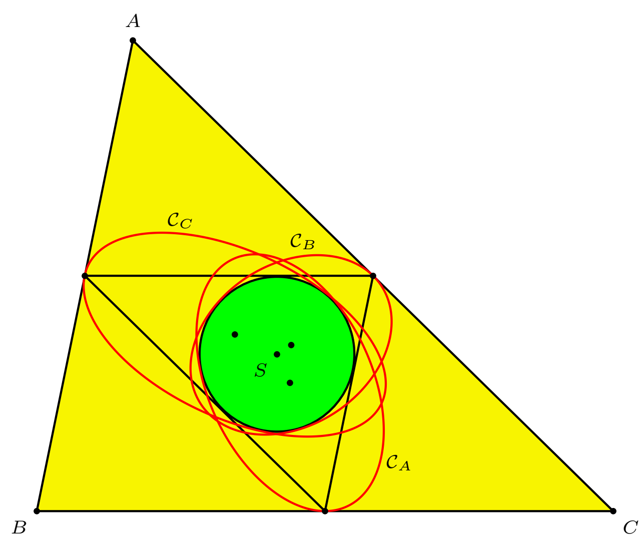

Figure 1

For example, if $\mathcal{C}$ is the incircle of the medial triangle, the conic $\mathcal{C}_A$ is tangent to BC at its midpoint, and contains the points $(s : s-b : b)$, $(s : c : s-c)$,

[1]By Chasles' theorem on polarity [1, 5.61], each triangle is perspective to its polar triangle. The perspector is called the perspector of the conic.

$((a+b+c)(b+c-a) : 2c(c+a-b) : b^2+3c^2-a^2+2ca)$ and $((a+b+c)(b+c-a) : 3b^2+c^2-a^2+2ab : 2b(a+b-c))$. It has center $(s : c+a : a+b)$. See Figure 1.

The other possibility for a double point is when $\mathcal{C}_A$ degenerates into ℓ_A. To investigate this case we prove the following proposition.

Proposition 2. *If $\mathcal{C}_A$ degenerates into a line, the triangle ABC is selfpolar with respect to each conic tangent to the sides of two cevian triangles. The cevian triangle of the trilinear pole of any tangent to such a conic is tritangent to this conic.*

Proof. Let P be a point and $A^PB^PC^P$ its anticevian triangle. ABC is a polar triangle with respect to each conic through $A^PB^PC^P$, as ABC are the diagonal points of the complete quadrilateral $PA^PB^PC^P$. Now consider a second anticevian triangle $A^QB^QC^Q$ of Q. The vertices of $A^PB^PC^P$ and $A^QB^QC^Q$ lie on a conic[2] $\mathcal{K}$. But we also know that triangle PB^PC^P is the anticevian triangle of A^P. So PB^PC^P and $A^QB^QC^Q$ lie on a conic as well, and having 5 common points this must be $\mathcal{K}$. We conclude that ABC is selfpolar with respect to $\mathcal{K}$.

Let R be a point on $\mathcal{K}$. AR intersects $\mathcal{K}$ in a second point R'. Let R_A be the intersection AR and BC, then R and R' are harmonic with respect to A and R_A. But that means that $R' = A^R$ is the $A-$vertex of the anti-cevian triangle of R. Consequently the anti-cevian triangle of R lies on $\mathcal{K}$. Proposition 2 is now proved by duality. □

In the proof B^PC^P is the side of two anticevian triangles inscribed in $\mathcal{K}$ - by duality this means that the vertex of a cevian triangle tangent to $\mathcal{K}$ is a common vertex of two such cevian triangles. In the case of ℓ_A intersecting BC in a double point, clearly the two triangles are cevian triangles with respect to the triangle bounded by AB, AC and ℓ_A. Were they cevian triangles also with respect to ABC, then the four sidelines of these cevian triangles would form the dual of an anticevian triangle, and ABC would be selfpolar with respect to $\mathcal{C}$, and ℓ_A would be BC.

We conclude that two distinct triangles inscribed in ABC and circumscribing $\mathcal{C}$ cannot be cevian triangles.

In the case ABC is selfpolar with respect to $\mathcal{C}$, so that $\mathcal{C}_A$ degenerates into ℓ_A, not each point on ℓ_A belongs to (real) cevian triangles. On the other hand clearly infinitely many points on ℓ_A will lead to two cevian triangles tritangent to $\mathcal{C}$. The perpsectors run through a quartic, the tripoles of the tangents to $\mathcal{C}$.

Theorem 3. *Given a triangle ABC and a conic $\mathcal{C}$, the* triangle-conic poristic *triangles inscribed in ABC and tritangent to $\mathcal{C}$ are as follows.*

(i) *There are be no triangle-conic poristic triangle.*

(ii) *$\mathcal{C}$ is a conic inscribed in a cevian triangle, and ABC is not self-polar with respect to $\mathcal{C}$. In this case the cevian triangle is the only triangle-conic poristic triangle.*

[2] This follows from the dual of the well known theorem that two cevian triangles are circumscribed by and inscribed in a conic.

(iii) ABC *is self-polar with respect to* $\mathcal{C}$*. In this case there are infinitely many triangle-conic poristic triangles.*

(iv) *There are two distinct triangle-conic poristic triangles, which are not cevian triangles.*

Remarks. (1) In case of a conic with respect to which ABC is self-polar, instead of cevian triangles tritangent to $\mathcal{C}$, we should speak of cevian fourlines quadritangent to $\mathcal{C}$.

(2) When we investigate triangles inscribed in a conic and circumscribed to ABC we get similar results as Theorem 3, simply by duality.

In case $\mathcal{C}$ is a conic with respect to which ABC is selfpolar, we see that each tangent to $\mathcal{C}$ belongs to two cevian triangles tritangent to $\mathcal{C}$ and that each point on $\mathcal{C}$ belongs to two anticevian triangles inscribed in $\mathcal{C}$. In this case speak of *triangle-conic porism* and *conic-triangle porism* in extension of the well known Poncelet porism.

As an example, we consider the *nine-point circle triangles*, hence the medial and orthic triangles. We know that these circumscribe a conic $\mathcal{C}_N$, with respect to which ABC is selfpolar. By Proposition 2 we know that the perspectrices of the medial and orthic triangles are tangent to $\mathcal{C}_N$ as well, hence $\mathcal{C}_N$ must be a parabola tangent to the orthic axis. The barycentric equation of this parabola is

$$\frac{x^2}{a^2(b^2-c^2)} + \frac{y^2}{b^2(c^2-a^2)} + \frac{z^2}{c^2(a^2-b^2)} = 0.$$

Its focus is X_{115} of [3, 4], its directrix the Brocard axis, and its axis is the Simson line of X_{98}. See Figure 2. The parabola contains the infinite point X_{512} and passes through X_{661}, X_{647} and X_{2519}. The Brianchon point of the parabola with respect to the medial triangle is $X_{670}(\text{medial})$.

The perspectors of the tangent cevian triangles run through the quartic

$$a^2(b^2-c^2)y^2z^2 + b^2(c^2-a^2)z^2x^2 + c^2(a^2-b^2)x^2y^2 = 0,$$

which is the isotomic conjugate of the conic

$$a^2(b^2-c^2)x^2 + b^2(c^2-a^2)y^2 + c^2(a^2-b^2)z^2 = 0$$

through the vertices of the antimedial triangle, the centroid, and the isotomic conjugates of the incenter and the orthocenter.

This special case leads us to amusing consequences, to which we were pointed by [2]. The sides of every cevian triangle and its perspectrix are tangent to one parabola inscribed in the medial triangle. Consequently the *isotomic conjugates*[3] with respect to to the medial triangle of these are parallel.

In the dual case, we conclude for instance that the isotomic conjugates with respect to the antimedial triangle of the vertices and perspector D of any anticevian triangle are collinear with the centroid G. The line is GD', where D' is the barycentric square of D.

[3]The isotomic conjugate of a line ℓ with respect to a triangle is the line passing through the intercepts of ℓ with the sides reflected through the corresponding midpoints. In [3] this is referred to as *isotomic transversal*.

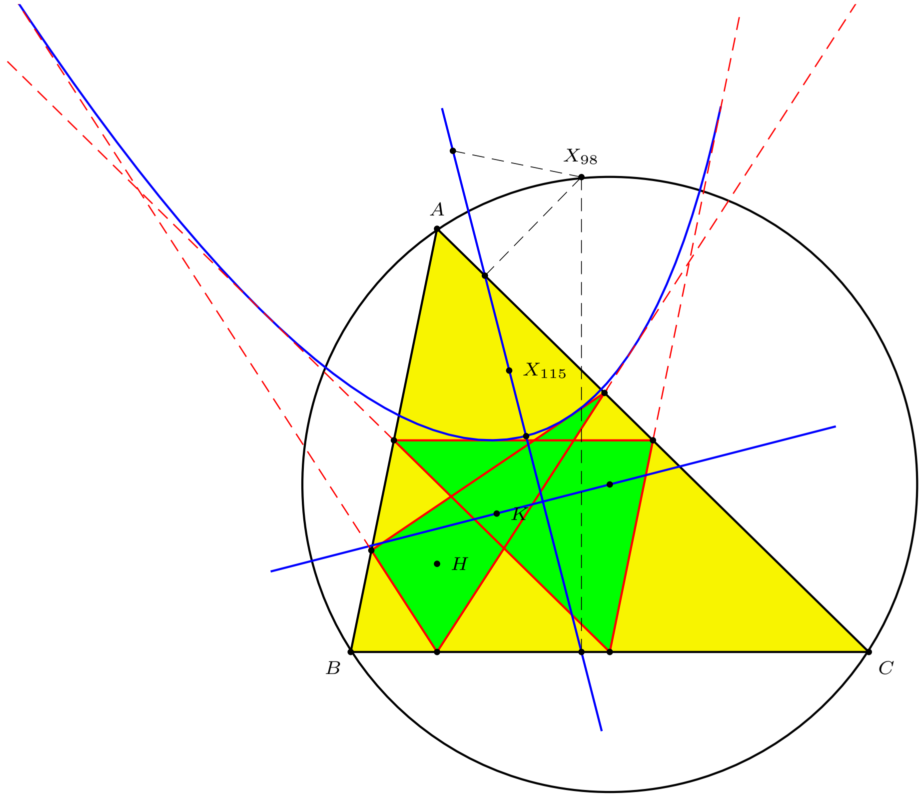

Figure 2.

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