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A Maximal Property of Cyclic Quadrilaterals

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Abstract. We give a very simple proof of the well known fact that among all quadrilaterals with given side lengths, the cyclic one has maximal area.

Among all quadrilaterals $ABCD$ be with given side lengths $AB = a$, $BC = b$, $CD = c$, $DA = d$, it is well known that the one with greatest area is the cyclic quadrilateral. All known proofs of this result make use of Brahmagupta formula. See, for example, [1, p.50]. In this note we give a very simple geometric proof.

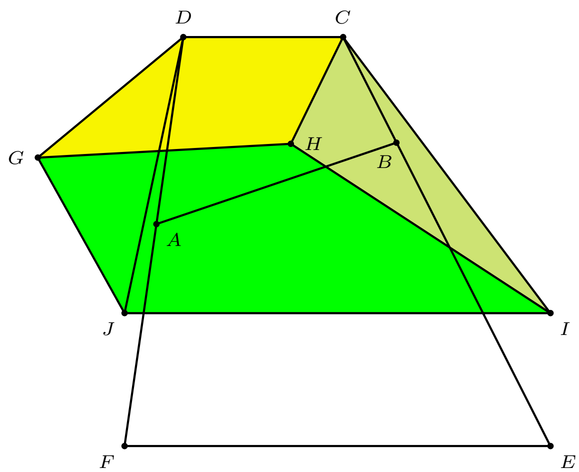

Figure 1

Let $ABCD$ be the cyclic quadrilateral and $GHCD$ an arbitrary one with the same side lengths: $GH = a$, $HC = b$, $CD = c$ and $DG = d$. Construct quadrilaterals $EFAB$ similar to $ABCD$ and $IJGH$ similar to $GHCD$ (in the same order of vertices). Note that

(i) FE is parallel to DC since $ABCD$ is cyclic and DAF, CBE are straight lines;
(ii) JI is also parallel to DC since

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$$\begin{aligned}
\angle CDJ + \angle DJI =& (\angle CDG - \angle JDG) + (\angle GJI - \angle GJD) \\
=& (\angle CDG - \angle JDG) + (\angle CHG - \angle GJD) \\
=& \angle CDG + \angle CHG - (\angle JDG + \angle GJD) \\
=& \angle CDG + \angle CHG - (180^\circ - \angle DGJ) \\
=& \angle CDG + \angle CHG + (\angle DGH + \angle HGJ) - 180^\circ \\
=& \angle CDG + \angle CHG + \angle DGH + \angle HCD - 180^\circ \\
=& 180^\circ.
\end{aligned}$$

Since the ratios of similarity of the quadrilaterals are both $\frac{a}{c}$, the areas of $ABEF$ and $GHIJ$ are $\frac{a^2}{c^2}$ times those of $ABCD$ and $GHCD$ respectively. It is enough to prove that

$$\text{area}(DCEF) \geq \text{area}(DCHIJGD).$$

In fact, since $GD \cdot GJ = HC \cdot HI$ and $\angle DGJ = \angle CHI$, it follows that area(DGJ) = area(CHI), and we have

$$\text{area}(DCHIJGD) = \text{area}(DCHG) + \text{area}(GHIJ) = \text{area}(DCIJ).$$

Note that

$$\begin{aligned}
\overrightarrow{CD} \cdot \overrightarrow{DJ} =& \overrightarrow{CD} \cdot (\overrightarrow{DG} + \overrightarrow{GJ}) \\
=& \overrightarrow{CD} \cdot \overrightarrow{DG} + \overrightarrow{CD} \cdot \overrightarrow{GJ} \\
=& \overrightarrow{CD} \cdot \overrightarrow{DG} + \frac{c^2}{a^2}\left(\overrightarrow{IJ} \cdot \overrightarrow{GJ}\right) \\
=& \overrightarrow{CD} \cdot \overrightarrow{DG} - \overrightarrow{CH} \cdot \overrightarrow{HG} \\
=& \frac{1}{2}\left(a^2 + b^2 - CG^2\right) - \frac{1}{2}\left(c^2 + d^2 - CG^2\right) \\
=& \frac{1}{2}(a^2 + b^2 - c^2 - d^2)
\end{aligned}$$

is independent of the position of J. This means that the line JF is perpendicular to DC; so is IE for a similar reason. The vector $\overrightarrow{DJ} = \overrightarrow{DG} + \overrightarrow{GJ}$ has a constant projection on $\overrightarrow{CD}$ (the same holds for $\overrightarrow{CI}$). We conclude that trapezium $DCEF$ has the greatest altitude among all these trapezia constructed the same way as $DCIJ$. Since all these trapezia have the same bases, $DCEF$ has the greatest area. This completes the proof that among quadrilaterals of given side lengths, the cyclic one has greatest area.

Reference

[1] N. D. Kazarinoff, *Geometric Inequalities*, Yale University, 1961.

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