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Elegant Geometric Constructions

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Dedicated to Professor M. K. Siu

Abstract. With the availability of computer software on dynamic geometry, beautiful and accurate geometric diagrams can be drawn, edited, and organized efficiently on computer screens. This new technological capability stimulates the desire to strive for elegance in actual geometric constructions. The present paper advocates a closer examination of the geometric meaning of the algebraic expressions in the analysis of a construction problem to actually effect a construction as elegantly and efficiently as possible on the computer screen. We present a fantasia of euclidean constructions the analysis of which make use of elementary algebra and very basic knowledge of euclidean geometry, and focus on incorporating simple algebraic expressions into actual constructions using the Geometer's Sketchpad®.

After a half century of curriculum reforms, it is fair to say that mathematicians and educators have come full circle in recognizing the relevance of Euclidean geometry in the teaching and learning of mathematics. For example, in [15], J. E. McClure reasoned that "Euclidean geometry is the only mathematical subject that is really in a position to provide the grounds for its own axiomatic procedures". See also [19]. Apart from its traditional role as the training ground for logical reasoning, Euclidean geometry, with its construction problems, provides a stimulating milieu of learning mathematics *constructivistically*. One century ago, D. E. Smith [17, p.95] explained that the teaching of constructions using ruler and compass serves several purposes: "it excites [students'] interest, it guards against the slovenly figures that so often lead them to erroneous conclusions, it has a genuine value for the future artisan, and its shows that geometry is something besides mere theory". Around the same time, the British Mathematical Association [16] recommended teaching school geometry as two parallel courses of *Theorems* and *Constructions*. "The course of constructions should be regarded as a *practical*

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course, the constructions being accurately made with instruments, and no construction, or proof of a construction, should be deemed invalid by reason of its being different from that given in Euclid, or by reason of its being based on theorems which Euclid placed after it".

A good picture is worth more than a thousand words. This is especially true for students and teachers of geometry. With good illustrations, concepts and problems in geometry become transparent and more understandable. However, the difficulty of drawing good blackboard geometric sketches is well appreciated by every teacher of mathematics. It is also true that many interesting problems on constructions with ruler and compass are genuinely difficult and demand great insights for solution, as in the case of geometrical proofs. Like handling difficult problems in synthetic geometry with analytic geometry, one analyzes construction problems by the use of algebra. It is well known that historically analysis of such ancient construction problems as the trisection of an angle and the duplication of the cube gave rise to the modern algebraic concept of field extension. A geometric construction can be effected with ruler and compass if and only if the corresponding algebraic problem is reducible to a sequence of linear and quadratic equations with constructible coefficients. For all the strength and power of such algebraic analysis of geometric problems, it is often impractical to carry out detailed constructions with paper and pencil, so much so that in many cases one is forced to settle for mere constructibility. For example, Howard Eves, in his solution [6] of the problem of construction of a triangle given the lengths of a side and the median and angle bisector on the same side, made the following remark after proving constructibility.

> The devotee of the game of Euclidean constructions is not really interested in the actual mechanical construction of the sought triangle, but merely in the assurance that the construction is possible. To use a phrase of Jacob Steiner, the devotee performs his construction "simply by means of the tongue" rather than with actual instruments on paper.

Now, the availability in recent years of computer software on dynamic geometry has brought about a change of attitude. Beautiful and accurate geometric diagrams can be drawn, edited, and organized efficiently on computer screens. This new technological capability stimulates the desire to strive for elegance in actual geometric constructions. The present paper advocates a closer examination of the geometric meaning of the algebraic expressions in the analysis of a construction problem to actually effect a construction as elegantly and efficiently as possible on the computer screen. [1] We present a fantasia of euclidean constructions the analysis of which make use of elementary algebra and very basic knowledge of euclidean geometry. [2] We focus on incorporating simple algebraic expressions into actual constructions using the Geometer's Sketchpad®. The tremendous improvement

[1]See §6.1 for an explicit construction of the triangle above with a given side, median, and angle bisector.

[2]The Geometer's Sketchpad® files for the diagrams in this paper are available from the author's website http://www.math.fau.edu/yiu/Geometry.html.

on the economy of time and effort is hard to exaggerate. The most remarkable feature of the Geometer's Sketchpad® is the capability of customizing a tool folder to make constructions as efficiently as one would like. Common, basic constructions need only be performed once, and saved as tools for future use. We shall use the Geometer's Sketchpad® simply as ruler and compass, assuming a tool folder containing at least the following tools[3] for ready use:
(i) basic shapes such as equilateral triangle and square,
(ii) tangents to a circle from a given point,
(iii) circumcircle and incircle of a triangle.

Sitting in front of the computer screen trying to perform geometric constructions is a most ideal constructivistic learning environment: a student is to bring his geometric knowledge and algebraic skill to bear on natural, concrete but challenging problems, experimenting with various geometric interpretations of concrete algebraic expressions. Such analysis and explicit constructions provide a fruitful alternative to the traditional emphasis of the deductive method in the learning and teaching of geometry.

1. Some examples

We present a few examples of constructions whose elegance is suggested by an analysis a little more detailed than is necessary for constructibility or routine constructions. A number of constructions in this paper are based on diagrams in the interesting book [9]. We adopt the following notation for circles:
(i) $A(r)$ denotes the circle with center A, radius r;
(ii) $A(B)$ denotes the circle with center A, passing through the point B, and
(iii) (A) denotes a circle with center A and unspecified radius, but unambiguous in context.

1.1. Construct a regular octagon by cutting corners from a square.

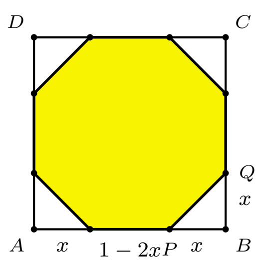

Figure 1A

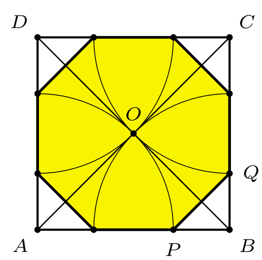

Figure 1B

Suppose an isosceles right triangle of (shorter) side x is to be cut from each corner of a unit square to make a regular octagon. See Figure 1A. A simple calculation shows that $x = 1 - \frac{\sqrt{2}}{2}$. This means $AP = 1 - x = \frac{\sqrt{2}}{2}$. The point P, and the

[3]A construction appearing in sans serif is assumed to be one readily performable with a customized tool.

other vertices, can be easily constructed by intersecting the sides of the square with quadrants of circles with centers at the vertices of the square and passing through the center O. See Figure 1B.

1.2. The centers A and B of two circles lie on the other circle. Construct a circle tangent to the line AB, to the circle (A) internally, and to the circle (B) externally.

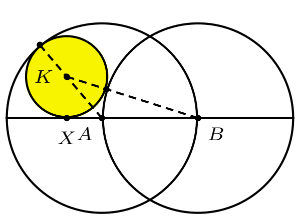

Figure 2A

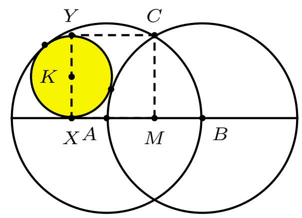

Figure 2B

Suppose $AB = a$. Let r = radius of the required circle (K), and $x = AX$, where X is the projection of the center K on the line AB. We have

$$(a+r)^2 = r^2 + (a+x)^2, \qquad (a-r)^2 = r^2 + x^2.$$

Subtraction gives $4ar = a^2 + 2ax$ or $x + \frac{a}{2} = 2r$. This means that in Figure 2B, $CMXY$ is a square, where M is the midpoint of AB. The circle can now be easily constructed by first erecting a square on CM.

1.3. *Equilateral triangle in a rectangle.* Given a rectangle $ABCD$, construct points P and Q on BC and CD respectively such that triangle APQ is equilateral.

Construction 1. *Construct* equilateral triangles CDX *and* BCY*, with* X *and* Y *inside the rectangle. Extend* AX *to intersect* BC *at* P *and* AY *to intersect* CD *at* Q.

The triangle APQ *is equilateral.* See Figure 3B.

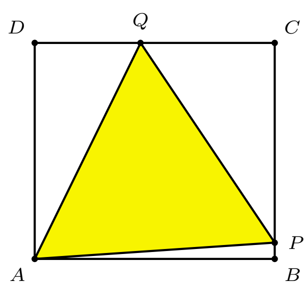

Figure 3A

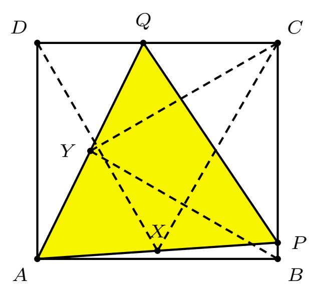

Figure 3B

This construction did not come from a lucky insight. It was found by an analysis. Let $AB = DC = a$, $BC = AD = b$. If $BP = y$, $DQ = x$ and APQ is equilateral, then a calculation shows that $x = 2a - \sqrt{3}b$ and $y = 2b - \sqrt{3}a$. From these expressions of x and y the above construction was devised.

1.4. *Partition of an equilateral triangle into 4 triangles with congruent incircles.* Given an equilateral triangle, construct three lines each through a vertex so that the incircles of the four triangles formed are congruent. See Figure 4A and [9, Problem 2.1.7] and [10, Problem 5.1.3], where it is shown that if each side of the equilateral triangle has length a, then the small circles all have radii $\frac{1}{8}(\sqrt{7}-\sqrt{3})a$. Here is a calculation that leads to a very easy construction of these lines.

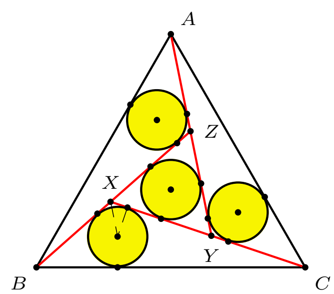

Figure 4A

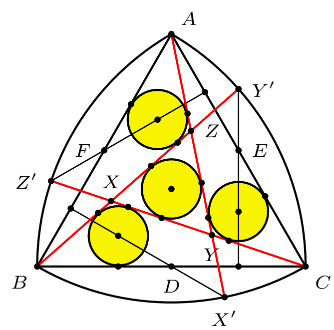

Figure 4B

In Figure 4A, let $CX = AY = BZ = a$ and $BX = CY = AZ = b$. The equilateral triangle XYZ has sidelength $a-b$ and inradius $\frac{\sqrt{3}}{6}(a-b)$. Since $\angle BXC = 120^\circ$, $BC = \sqrt{a^2+ab+b^2}$, and the inradius of triangle BXC is

$$\frac{1}{2}(a+b-\sqrt{a^2+ab+b^2})\tan 60^\circ = \frac{\sqrt{3}}{2}(a+b-\sqrt{a^2+ab+b^2}).$$

These two inradii are equal if and only if $3\sqrt{a^2+ab+b^2} = 2(a+2b)$. Applying the law of cosines to triangle XBC, we obtain

$$\cos XBC = \frac{(a^2+ab+b^2)+b^2-a^2}{2b\sqrt{a^2+ab+b^2}} = \frac{a+2b}{2\sqrt{a^2+ab+b^2}} = \frac{3}{4}.$$

In Figure 4B, Y' is the intersection of the arc $B(C)$ and the perpendicular from the midpoint E of CA to BC. The line BY' makes an angle $\arccos\frac{3}{4}$ with BC. The other two lines AX' and CZ' are similarly constructed. These lines bound the equilateral triangle XYZ, and the four incircles can be easily constructed. Their centers are simply the reflections of X' in D, Y' in E, and Z' in F.

2. Some basic constructions

2.1. *Geometric mean and the solution of quadratic equations.* The following constructions of the geometric mean of two lengths are well known.

Construction 2. (a) *Given two segments of length a, b, mark three points A, P, B on a line (P between A and B) such that $PA = a$ and $PB = b$. Describe a semicircle with AB as diameter, and let the perpendicular through P intersect the semicircle at Q. Then $PQ^2 = AP \cdot PB$, so that the length of PQ is the geometric mean of a and b.* See Figure 5A.

(b) *Given two segments of length $a < b$, mark three points P, A, B on a line such that $PA = a$, $PB = b$, and A, B are on the* same *side of P. Describe a semicircle with PB as diameter, and let the perpendicular through A intersect the semicircle at Q. Then $PQ^2 = PA \cdot PB$, so that the length of PQ is the geometric mean of a and b.* See Figure 5B.

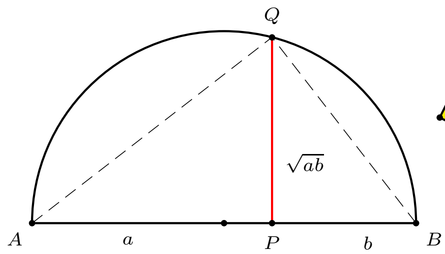

Figure 5A

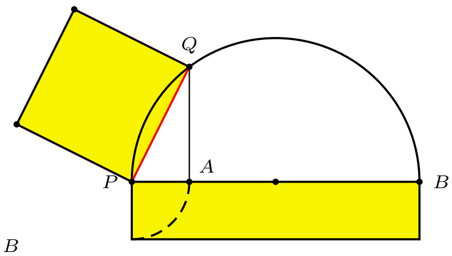

Figure 5B

More generally, a quadratic equation can be solved by applying the theorem of intersecting chords: *If a line through P intersects a circle $O(r)$ at X and Y, then the product $PX \cdot PY$ (of signed lengths) is equal to $OP^2 - r^2$.* Thus, if two chords AB and XY intersect at P, then $PA \cdot PB = PX \cdot PY$. See Figure 6A. In particular, if P is outside the circle, and if PT is a tangent to the circle, then $PT^2 = PX \cdot PY$ for any line intersecting the circle at X and Y. See Figure 6B.

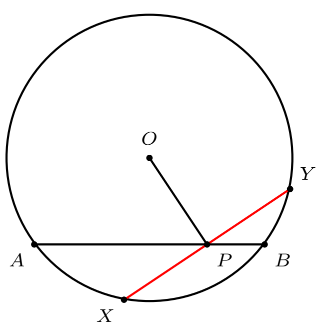

Figure 6A

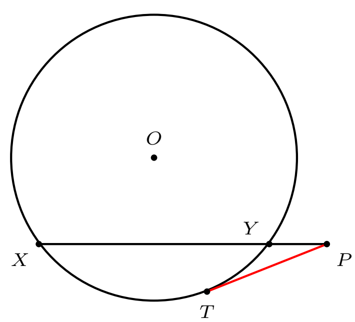

Figure 6B

A quadratic equation can be put in the form $x(x \pm a) = b^2$ or $x(a - x) = b^2$. In the latter case, for real solutions, we require $b \leq \frac{a}{2}$. If we arrange a and b as the legs of a right triangle, then the positive roots of the equation can be easily constructed as in Figures 6C and 6D respectively.

The algebraic method of the solution of a quadratic equation by completing squares can be easily incorporated geometrically by using the Pythagorean theorem. We present an example.

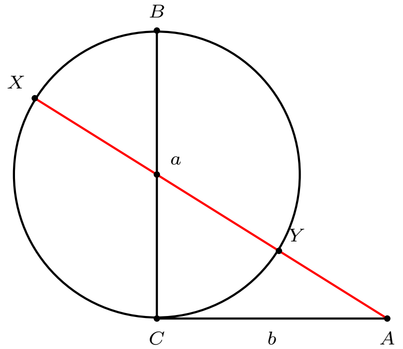

Figure 6C

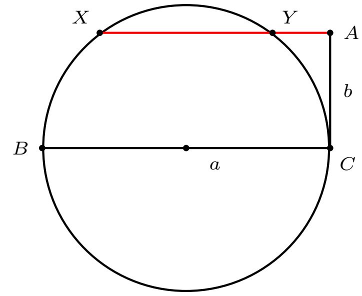

Figure 6D

2.1.1. Given a chord BC perpendicular to a diameter XY of circle (O), to construct a line through X which intersects the circle at A and BC at T such that AT has a given length t. Clearly, $t \leq YM$, where M is the midpoint of BC.

Let $AX = x$. Since $\angle CAX = \angle CYX = \angle TCX$, the line CX is tangent to the circle ACT. It follows from the theorem of intersecting chords that $x(x-t) = CX^2$. The method of completing squares leads to

$$x = \frac{t}{2} + \sqrt{CX^2 + \left(\frac{t}{2}\right)^2}.$$

This suggests the following construction.[4]

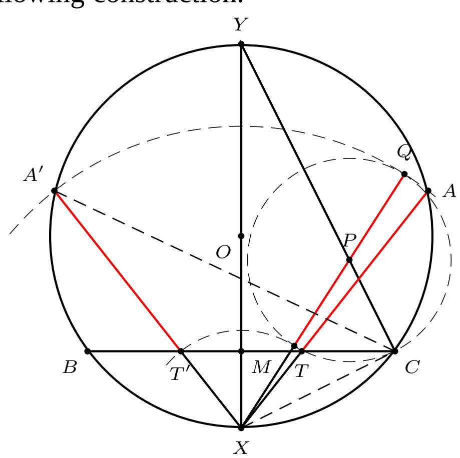

Figure 7

Construction 3. *On the segment CY, choose a point P such that $CP = \frac{t}{2}$. Extend XP to Q such that $PQ = PC$. Let A be an intersection of $X(Q)$ and (O). If the line XA intersects BC at T, then $AT = t$.* See Figure 7.

[4] This also solves the construction problem of triangle ABC with given angle A, the lengths a of its opposite side, and of the bisector of angle A.

2.2. *Harmonic mean and the equation* $\frac{1}{a} + \frac{1}{b} = \frac{1}{t}$. The harmonic mean of two quantities a and b is $\frac{2ab}{a+b}$. In a trapezoid of parallel sides a and b, the parallel through the intersection of the diagonals intercepts a segment whose length is the harmonic mean of a and b. See Figure 8A. We shall write this harmonic mean as $2t$, so that $\frac{1}{a} + \frac{1}{b} = \frac{1}{t}$. See Figure 8B.

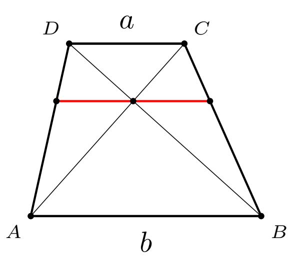

Figure 8A

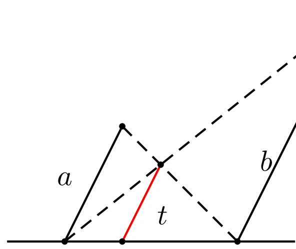

Figure 8B

Here is another construction of t, making use of the formula for the length of an angle bisector in a triangle. If $BC = a$, $AC = b$, then the angle bisector CZ has length

$$t_c = \frac{2ab}{a+b} \cos \frac{C}{2} = 2t \cos \frac{A}{2}.$$

The length t can therefore be constructed by completing the rhombus $CXZY$ (by constructing the perpendicular bisector of CZ to intersect BC at X and AC at Y). See Figure 9A. In particular, if the triangle contains a right angle, this trapezoid is a square. See Figure 9B.

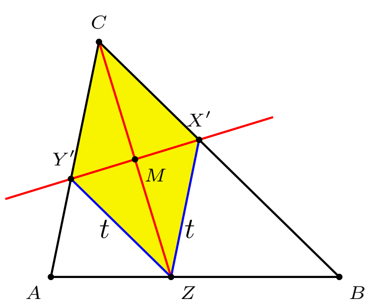

Figure 9A

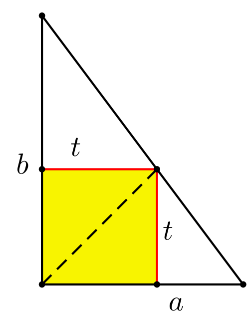

Figure 9B

3. The shoemaker's knife

3.1. *Archimedes' Theorem.* A shoemaker's knife (or arbelos) is the region obtained by cutting out from a semicircle with diameter AB the two smaller semicircles with diameters AP and PB. Let $AP = 2a$, $PB = 2b$, and the common tangent of the smaller semicircles intersect the large semicircle at Q. The following remarkable theorem is due to Archimedes. See [12].

Theorem 1 (Archimedes)**.** (1) *The two circles each tangent to PQ, the large semicircle and one of the smaller semicircles have equal radii $t = \frac{ab}{a+b}$.* See Figure 10A.

(2) *The circle tangent to each of the three semicircles has radius*

$$\rho = \frac{ab(a+b)}{a^2+ab+b^2}. \tag{1}$$

See Figure 10B.

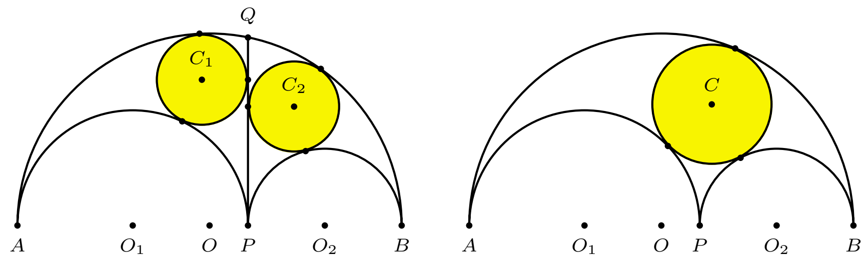

Figure 10A Figure 10B

Here is a simple construction of the Archimedean "twin circles". Let Q_1 and Q_2 be the "highest" points of the semicircles $O_1(a)$ and $O_2(b)$ respectively. The intersection $C_3 = O_1Q_2 \cap O_2Q_1$ is a point "above" P, and $C_3P = t = \frac{ab}{a+b}$.

Construction 4. *Construct the circle $P(C_3)$ to intersect the diameter AB at P_1 and P_2 (so that P_1 is on AP and P_2 is on PB).*

The center C_1 (respectively C_2) is the intersection of the circle $O_1(P_2)$ (respectively $O_2(P_1)$) and the perpendicular to AB at P_1 (respectively P_2). See Figure 11.

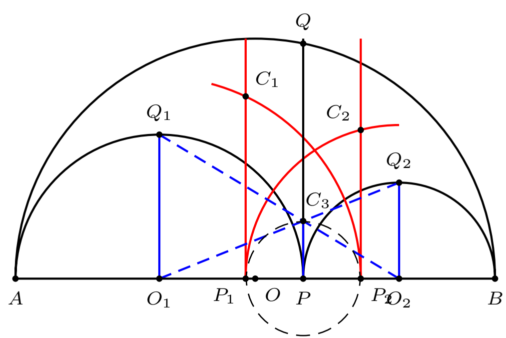

Figure 11

Theorem 2 (Bankoff [3])**.** *If the incircle $C(\rho)$ of the shoemaker's knife touches the smaller semicircles at X and Y, then the circle through the points P, X, Y has the same radius t as the Archimedean circles.* See Figure 12.

This gives a very simple construction of the incircle of the shoemaker's knife.

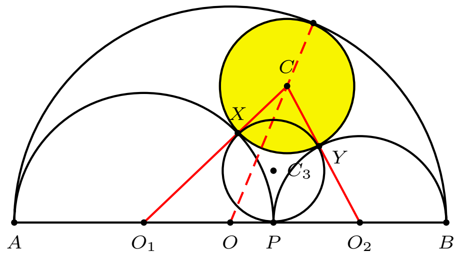

Figure 12

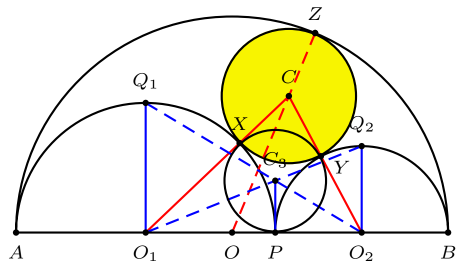

Figure 13

Construction 5. *Let $X = C_3(P) \cap O_1(a)$, $Y = C_3(P) \cap O_2(b)$, and $C = O_1X \cap O_2Y$. The circle $C(X)$ is the incircle of the shoemaker's knife. It touches the large semicircle at $Z = OC \cap O(a+b)$.* See Figure 13.

A rearrangement of (1) in the form

$$\frac{1}{a+b} + \frac{1}{\rho} = \frac{1}{t}$$

leads to another construction of the incircle (C) by directly locating the center and one point on the circle. See Figure 14.

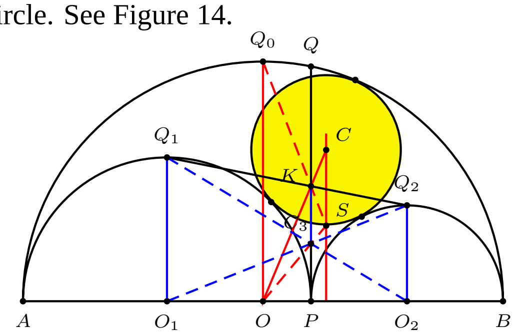

Figure 14

Construction 6. *Let Q_0 be the "highest" point of the semicircle $O(a+b)$. Construct*

(i) $K = Q_1Q_2 \cap PQ$,

(ii) $S = OC_3 \cap Q_0K$, *and*

(iii) *the perpendicular from S to AB to intersect the line OK at C.*

The circle $C(S)$ is the incircle of the shoemaker's knife.

3.2. *Other simple constructions of the incircle of the shoemaker's knife.* We give four more simple constructions of the incircle of the shoemaker's knife. The first is by Leon Bankoff [1]. The remaining three are by Peter Woo [21].

Construction 7 (Bankoff)**.** (1) *Construct the circle $Q_1(A)$ to intersect the semicircles $O_2(b)$ and $O(a+b)$ at X and Z respectively.*

(2) *Construct the circle $Q_2(B)$ to intersect the semicircles $O_1(a)$ and $O(a+b)$ at Y and the same point Z in* (1) *above.*

The circle through X, Y, Z *is the incircle of the shoemaker's knife.* See Figure 15.

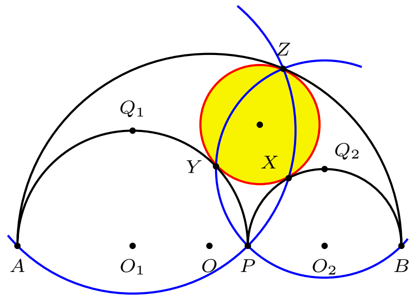

Figure 15

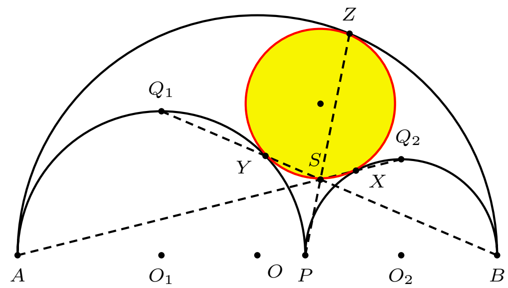

Figure 16

Construction 8 (Woo). (1) *Construct the line* AQ_2 *to intersect the semicircle* $O_2(b)$ *at* X.

(2) *Construct the line* BQ_1 *to intersect the semicircle* $O_1(a)$ *at* Y.

(3) *Let* $S = AQ_2 \cap BQ_1$. *Construct the line* PS *to intersect the semicircle* $O(a+b)$ *at* Z.

The circle through X, Y, Z *is the incircle of the shoemaker's knife.* See Figure 16.

Construction 9 (Woo). *Let* M *be the "lowest" point of the circle* $O(a+b)$. *Construct*

(i) *the circle* $M(A)$ *to intersect* $O_1(a)$ *at* Y *and* $O_2(b)$ *at* X,

(ii) *the line* MP *to intersect the semicircle* $O(a+b)$ *at* Z.

The circle through X, Y, Z *is the incircle of the shoemaker's knife.* See Figure 17.

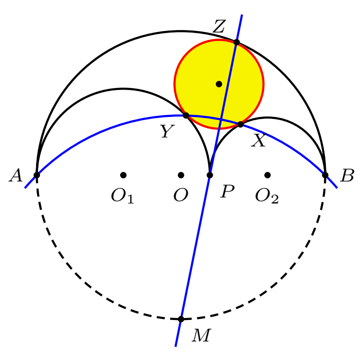

Figure 17

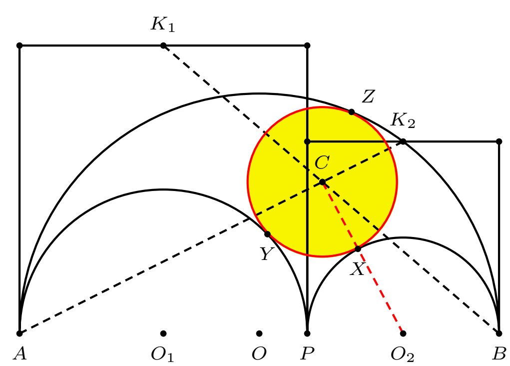

Figure 18

Construction 10 (Woo)**.** *Construct* squares *on AP and PB on the same side of the shoemaker knife. Let K_1 and K_2 be the midpoints of the opposite sides of AP and PB respectively. Let $C = AK_2 \cap BK_1$, and $X = CO_2 \cap O_2(b)$.*

The circle $C(X)$ is the incircle of the shoemaker's knife. See Figure 18.

4. Animation of bicentric polygons

A famous theorem of J. V. Poncelet states that if between two conics $\mathcal{C}_1$ and $\mathcal{C}_2$ there is a polygon of n sides with vertices on $\mathcal{C}_1$ and sides tangent to $\mathcal{C}_2$, then there is one such polygon of n sides with a vertex at an arbitrary point on $\mathcal{C}_1$. See, for example, [5]. For circles $\mathcal{C}_1$ and $\mathcal{C}_2$ and for $n = 3, 4$, we illustrate this theorem by constructing animation pictures based on simple metrical relations.

4.1. *Euler's formula.* Consider the construction of a triangle given its circumcenter O, incenter I and a vertex A. The circumcircle is $O(A)$. If the line AI intersects this circle again at X, then the vertices B and C are simply the intersections of the circles $X(I)$ and $O(A)$. See Figure 19A. This leads to the famous Euler formula

$$d^2 = R^2 - 2Rr, \tag{2}$$

where d is the distance between the circumcenter and the incenter.[5]

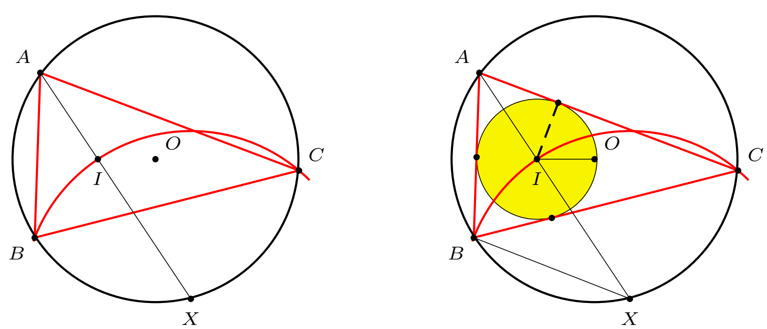

Figure 19A Figure 19B

4.1.1. Given a circle $O(R)$ and $r < \frac{R}{2}$, to construct a point I such that $O(R)$ and $I(r)$ are the circumcircle and incircle of a triangle.

Construction 11. *Let $P(r)$ be a circle tangent to (O) internally. Construct a line through O* tangent to the circle *$P(r)$ at a point I.*

The circle $I(r)$ is the incircle of triangles which have $O(R)$ as circumcircle. See Figure 20.

[5]*Proof*: If I is the incenter, then $AI = \frac{r}{\sin\frac{A}{2}}$ and $IX = IB = \frac{2R}{\sin\frac{A}{2}}$. See Figure 19B. The power of I with respect to the circumcircle is $d^2 - R^2 = IA \cdot IX = -r\sin\frac{A}{2} \cdot \frac{2R}{\sin\frac{A}{2}} = -2Rr$.

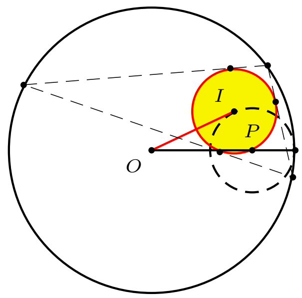

Figure 20

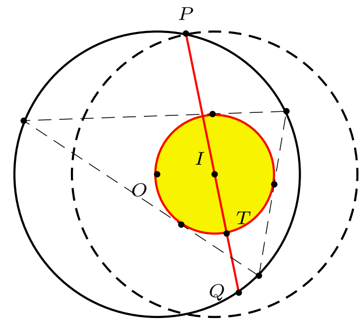

Figure 21

4.1.2. Given a circle $O(R)$ and a point I, to construct a circle $I(r)$ such that $O(R)$ and $I(r)$ are the circumcircle and incircle of a triangle.

Construction 12. *Construct the circle $I(R)$ to intersect $O(R)$ at a point P, and construct the line PI to intersect $O(R)$ again at Q. Let T be the midpoint of IQ.*

The circle $I(T)$ is the incircle of triangles which have $O(R)$ as circumcircle. See Figure 21.

4.1.3. Given a circle $I(r)$ and a point O, to construct a circle $O(R)$ which is the circumcircle of triangles with $I(r)$ as incircle. Since $R = r + \sqrt{r^2 + d^2}$ by the Euler formula (2), we have the following construction. See Figure 22.

Construction 13. *Let IP be a radius of $I(r)$ perpendicular to IO. Extend OP to a point A such that $PA = r$.*

The circle $O(A)$ is the circumcircle of triangles which have $I(r)$ as incircle.

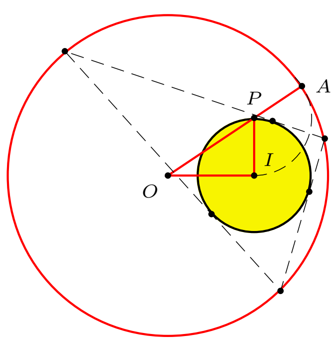

Figure 22

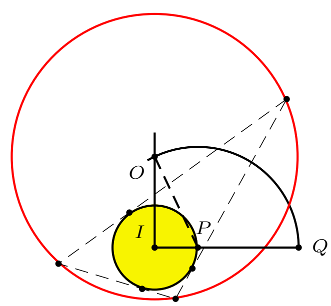

Figure 23

4.1.4. Given $I(r)$ and $R > 2r$, to construct a point O such that $O(R)$ is the circumcircle of triangles with $I(r)$ as incircle.

Construction 14. *Extend a radius IP to Q such that $IQ = R$. Construct the perpendicular to IP at I to intersect the circle $P(Q)$ at O.*

The circle $O(R)$ is the circumcircle of triangles which have $I(r)$ as incircle. See Figure 23.

4.2. *Bicentric quadrilaterals.* A bicentric quadrilateral is one which admits a circumcircle and an incircle. The construction of bicentric quadrilaterals is based on the Fuss formula

$$2r^2(R^2 + d^2) = (R^2 - d^2)^2, \tag{3}$$

where d is the distance between the circumcenter and incenter of the quadrilateral. See [7, §39].

4.2.1. Given a circle $O(R)$ and a point I, to construct a circle $I(r)$ such that $O(R)$ and $I(r)$ are the circumcircle and incircle of a quadrilateral.

The Fuss formula (3) can be rewritten as

$$\frac{1}{r^2} = \frac{1}{(R+d)^2} + \frac{1}{(R-d)^2}.$$

In this form it admits a very simple interpretation: r can be taken as the altitude on the hypotenuse of a right triangle whose shorter sides have lengths $R \pm d$. See Figure 24.

Construction 15. *Extend IO to intersect $O(R)$ at a point A. On the perpendicular to IA at I construct a point K such that $IK = R - d$. Construct the altitude IP of the right triange AIK.*

The circles $O(R)$ and $I(P)$ are the circumcircle and incircle of bicentric quadrilaterals.

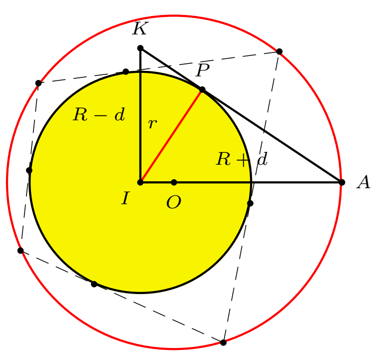

Figure 24

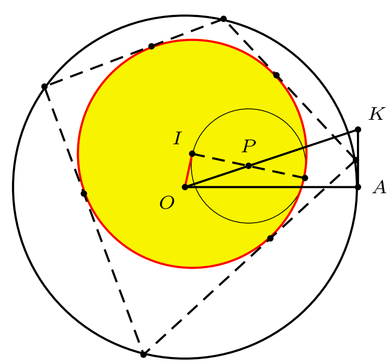

Figure 25

4.2.2. Given a circle $O(R)$ and a radius $r \leq \frac{R}{\sqrt{2}}$, to construct a point I such that $I(r)$ is the incircle of quadrilaterals inscribed in $O(R)$, we rewrite the Fuss formula (3) in the form

$$d^2 = \left(\sqrt{R^2 + \frac{r^2}{4}} - \frac{r}{2}\right)\left(\sqrt{R^2 + \frac{r^2}{4}} - \frac{3r}{2}\right).$$

This leads to the following construction. See Figure 25.

Construction 16. *Construct a right triangle OAK with a right angle at A, $OA = R$ and $AK = \frac{r}{2}$. On the hypotenuse OK choose a point P such that $KP = r$. Construct a* tangent from O to the circle $P(\frac{r}{2})$. *Let I be the point of tangency.*

The circles $O(R)$ and $I(r)$ are the circumcircle and incircle of bicentric quadrilaterals.

4.2.3. Given a circle $I(r)$ and a point O, to construct a circle (O) such that these two circles are respectively the incircle and circumcircle of a quadrilateral. Again, from the Fuss formula (3),

$$R^2 = \left(\sqrt{d^2 + \frac{r^2}{4}} + \frac{r}{2}\right)\left(\sqrt{d^2 + \frac{r^2}{4}} + \frac{3r}{2}\right).$$

Construction 17. *Let E be the midpoint of a radius IB perpendicular to OI. Extend the ray OE to a point F such that $EF = r$. Construct a* tangent OT to the circle $F\left(\frac{r}{2}\right)$. *Then OT is a circumradius.*

5. Some circle constructions

5.1. *Circles tangent to a chord at a given point.* Given a point P on a chord BC of a circle (O), there are two circles tangent to BC at P, and to (O) internally. The radii of these two circles are $\dfrac{BP \cdot PC}{2(R \pm h)}$, where h is the distance from O to BC. They can be constructed as follows.

Construction 18. *Let M be the midpoint of BC, and XY be the diameter perpendicular to BC. Construct*
(i) *the circle center P, radius MX to intersect the arc BXC at a point Q,*
(ii) *the line PQ to intersect the circle (O) at a point H,*
(iii) *the circle $P(H)$ to intersect the line perpendicular to BC at P at K* (*so that H and K are on the same side of BC*).

The circle with diameter PK is tangent to the circle (O). See Figure 26A.

Replacing X by Y in (i) *above we obtain the other circle tangent to BC at P and internally to (O).* See Figure 26B.

5.2. *Chain of circles tangent to a chord.* Given a circle (Q) tangent internally to a circle (O) and to a chord BC at a given point P, there are two neighbouring circles tangent to (O) and to the same chord. These can be constructed easily by observing that in Figure 27, the common tangent of the two circles cuts out a segment whose

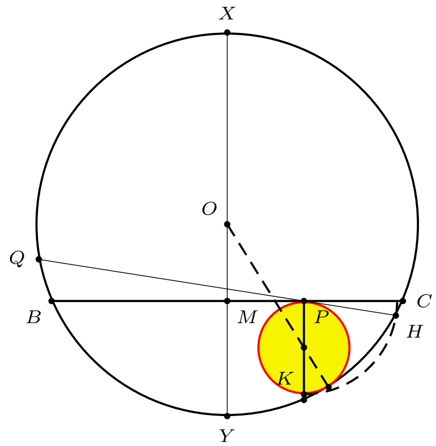

Figure 26A

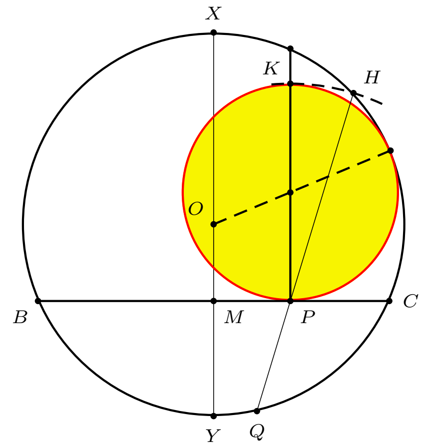

Figure 26B

midpoint is B. If (Q') is a neighbour of (Q), their common tangent passes through the midpoint M of the arc BC complementary to (Q). See Figure 28.

Construction 19. *Given a circle (Q) tangent to (O) and to the chord BC, construct*
(i) *the circle $M(B)$ to intersect (Q) at T_1 and T_2, MT_1 and MT_2 being tangents to (Q),*
(ii) *the bisector of the angle between MT_1 and BC to intersect the line QT_1 at Q_1.*

The circle $Q_1(T_1)$ is tangent to (O) and to BC.

Replacing T_1 by T_2 in (ii) *we obtain Q_2. The circle $Q_2(T_2)$ is also tangent to (O) and BC.*

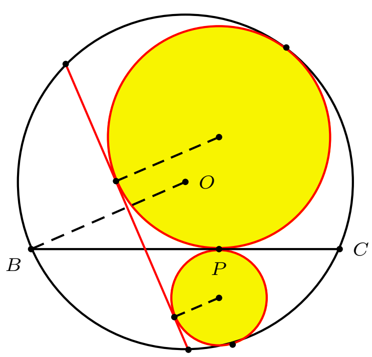

Figure 27

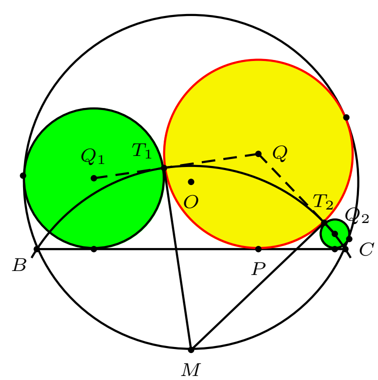

Figure 28

5.3. *Mixtilinear incircles.* Given a triangle ABC, we construct the circle tangent to the sides AB, AC, and also to the circumcircle internally. Leon Bankoff [4] called this the A- mixtilinear incircle of the triangle. Its center is clearly on the bisector of angle A. Its radius is $r \sec^2 \frac{A}{2}$, where r is the inradius of the triangle. The mixtilinear incircle can be constructed as follows. See Figure 29.

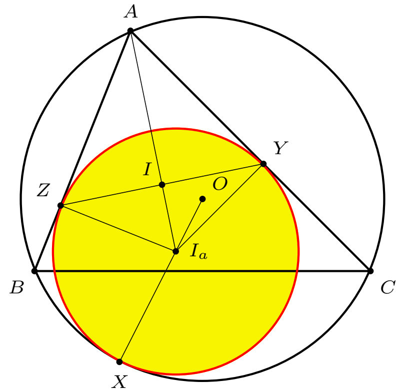

Figure 29

Construction 20 (Mixtilinear incircle)**.** *Let I be the* incenter *of triangle ABC. Construct*
(i) *the perpendicular to IA at I to intersect AC at Y,*
(ii) *the perpendicular to AY at Y to intersect the line AI at I_a.*
The circle $I_a(Y)$ is the A-mixtilinear incircle of ABC.

The other two mixtilinear incircles can be constructed in a similar way. For another construction, see [23].

5.4. *Ajima's construction.* The interesting book [10] by Fukagawa and Rigby contains a very useful formula which helps perform easily many constructions of inscribed circles which are otherwise quite difficult.

Theorem 3 (Ajima)**.** *Given triangles ABC with circumcircle (O) and a point P such that A and P are on the same side of BC, the circle tangent to the lines PB, PC, and to the circle (O) internally is the image of the incircle of triangle PBC under the homothety with center P and ratio $1 + \tan\frac{A}{2}\tan\frac{BPC}{2}$.*

Construction 21 (Ajima)**.** *Given two points B and C on a circle (O) and an arbitrary point P, construct*
(i) *a point A on (O) on the same side of BC as P,* (*for example, by taking the midpoint M of BC, and intersecting the ray MP with the circle (O)*)*,*
(ii) *the* incenter *I of triangle ABC,*
(iii) *the* incenter *I' of triangle PBC,*
(iv) *the perpendicular to $I'P$ at I' to intersect PC at Z.*
(v) *Rotate the ray ZI' about Z through an* (*oriented*) *angle equal to angle BAI to intersect the line AP at Q.*
Then the circle with center Q, tangent to the lines PB and PC, is also tangent to (O) internally. See Figure 30.

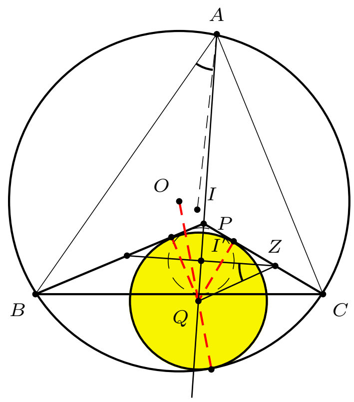

Figure 30

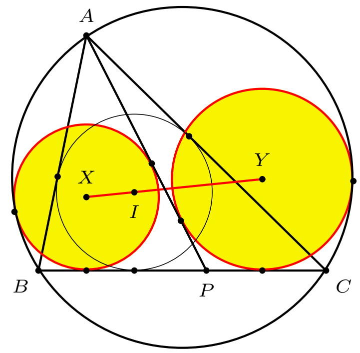

Figure 31

5.4.1. *Thébault's theorem.* With Ajima's construction, we can easily illustrate the famous Thébault theorem. See [18, 2] and Figure 31.

Theorem 4 (Thébault)**.** *Let P be a point on the side BC of triangle ABC. If the circles (X) and (Y) are tangent to AP, BC, and also internally to the circumcirle of the triangle, then the line XY passes through the incenter of the triangle.*

5.4.2. *Another example.* We construct an animation picture based on Figure 32 below. Given a segment AB and a point P, construct the squares $APX'X$ and $BPY'Y$ on the segments AP and BP. The locus of P for which A, B, X, Y are concyclic is the union of the perpendicular bisector of AB and the two quadrants of circles with A and B as endpoints. Consider P on one of these quadrants. The center of the circle $ABYX$ is the center of the other quadrant. Applying Ajima's construction to the triangle XAB and the point P, we easily obtain the circle tangent to AP, BP, and (O). Since $\angle APB = 135°$ and $\angle AXB = 45°$, the radius of this circle is twice the inradius of triangle APB.

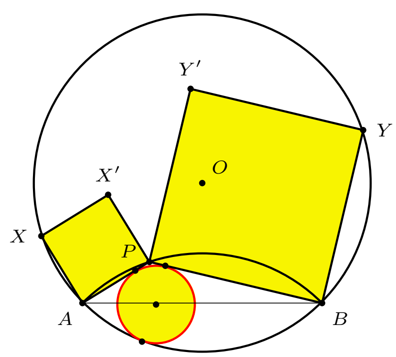

Figure 32

6. Some examples of triangle constructions

There is an extensive literature on construction problems of triangles with certain given elements such as angles, lengths, or specified points. Wernick [20] outlines a project of such with three given specific points. Lopes [14], on the other hand, treats extensively the construction problems with three given lengths such as sides, medians, bisectors, or others. We give three examples admitting elegant constructions. [6]

6.1. *Construction from a sidelength and the corresponding median and angle bisector.* Given the length $2a$ of a side of a triangle, and the lengths m and t of the median and the angle bisector on the same side, to construct the triangle. This is Problem 1054(a) of the *Mathematics Magazine* [6]. In his solution, Howard Eves denotes by z the distance between the midpoint and the foot of the angle bisector on the side $2a$, and obtains the equation

$$z^4 - (m^2 + t^2 + a^2)z^2 + a^2(m^2 - t^2) = 0,$$

from which he concludes constructibility (by ruler and compass). We devise a simple construction, assuming the data given in the form of a triangle $AM'T$ with $AT = t$, $AM' = m$ and $M'T = a$. See Figure 33. Writing $a^2 = m^2 + t^2 - 2tu$, and $z^2 = m^2 + t^2 - 2tw$, we simplify the above equation into

$$w(w - u) = \frac{1}{2}a^2. \tag{4}$$

Note that u is length of the projection of AM' on the line AT, and w is the length of the median AM on the bisector AT of the sought triangle ABC. The length w can be easily constructed, from this it is easy to complete the triangle ABC.

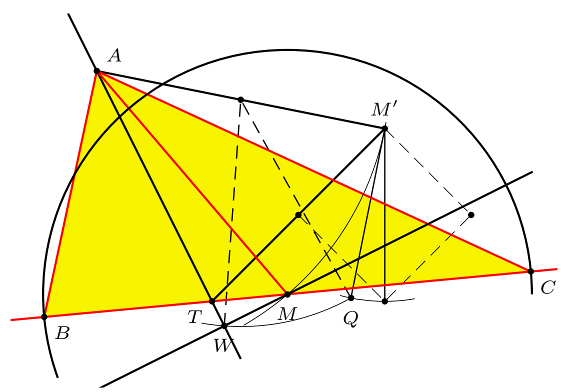

Figure 33

[6]Construction 3 (Figure 7) solves the construction problem of triangle ABC given angle A, side a, and the length t of the bisector of angle A. See Footnote 4.

Construction 22. (1) *On the perpendicular to AM' at M', choose a point Q such that $M'Q = \frac{M'T}{\sqrt{2}} = \frac{a}{\sqrt{2}}$.*

(2) *Construct the circle with center the midpoint of AM' to pass through Q and to intersect the line AT at W so that T and W are on the same side of A.* (*The length w of AW satisfies* (4) *above*).

(3) *Construct the perpendicular at W to AW to intersect the circle $A(M')$ at M.*

(4) *Construct the circle $M(a)$ to intersect the line MT at two points B and C.*

The triangle ABC has AT as bisector of angle A.

6.2. *Construction from an angle and the corresponding median and angle bisector.* This is Problem 1054(b) of the *Mathematics Magazine*. See [6]. It also appeared earlier as Problem E1375 of the *American Mathematical Monthly*. See [11]. We give a construction based on Thébault's solution.

Suppose the data are given in the form of a right triangle OAM, where $\angle AOM = A$ or $180° - A$, $\angle M = 90°$, $AM = m$, along with a point T on AM such that $AT = t$. See Figure 34.

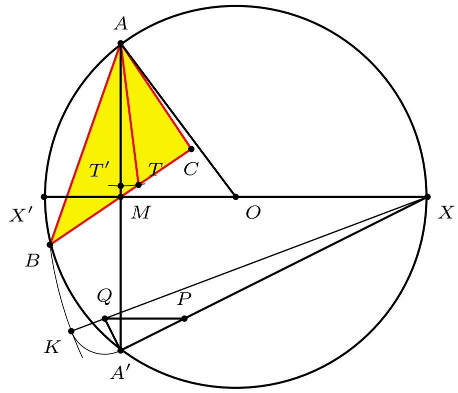

Figure 34

Construction 23. (1) *Construct the circle $O(A)$. Let A' be the mirror image of A in M. Construct the diameter XY perpendicular to AA', X the point for which $\angle AXA' = A$.*

(2) *On the segment $A'X$ choose a point P such that $A'P = \frac{t}{2}$. and construct the parallel through P to XY to intersect $A'Y$ at Q.*

(3) *Extend XQ to K such that $QK = QA'$.*

(4) *Construct a point B on $O(A)$ such that $XB = XK$, and its mirror image C in M.*

Triangle ABC has given angle A, median m and bisector t on the side BC.

6.3. *Construction from the incenter, orthocenter and one vertex.* This is one of the unsolved cases in Wernick [20]. See also [22]. Suppose we put the incenter I at the origin, $A = (a, b)$ and $H = (a, c)$ for $b > 0$. Let r be the inradius of the triangle.

A fairly straightforward calculation gives

$$r^2 - \frac{b-c}{2}r - \frac{1}{2}(a^2+bc) = 0. \tag{5}$$

If M is the midpoint of IA and P the orthogonal projection of H on the line IA, then $\frac{1}{2}(a^2+bc)$, being the dot product of IM and IH, is the (signed) product $IM \cdot IP$. Note that if angle AIH does not exceed a right angle, equation (5) admits a unique positive root. In the construction below we assume H closer than A to the perpendicular to AH through I.

Construction 24. *Given triangle AIH in which the angle AIH does not exceed a right angle, let M be the midpoint of IA, K the midpoint of AH, and P the orthogonal projection of H on the line IA.*
(1) *Construct the circle $\mathcal{C}$ through P, M and K. Let O be the center of $\mathcal{C}$ and Q the midpoint of PK.*
(2) *Construct a tangent from I to the circle $O(Q)$ intersecting $\mathcal{C}$ at T, with T farther from I than the point of tangency.*

The circle $I(T)$ is the incircle of the required triangle, which can be completed by constructing the tangents from A to $I(T)$, and the tangent perpendicular to AH through the "lowest" point of $I(T)$. See Figure 35.

If H is farther than A to the perpendicular from I to the line AH, the same construction applies, except that in (2) T is the intersection with $\mathcal{C}$ closer to I than the point of tangency.

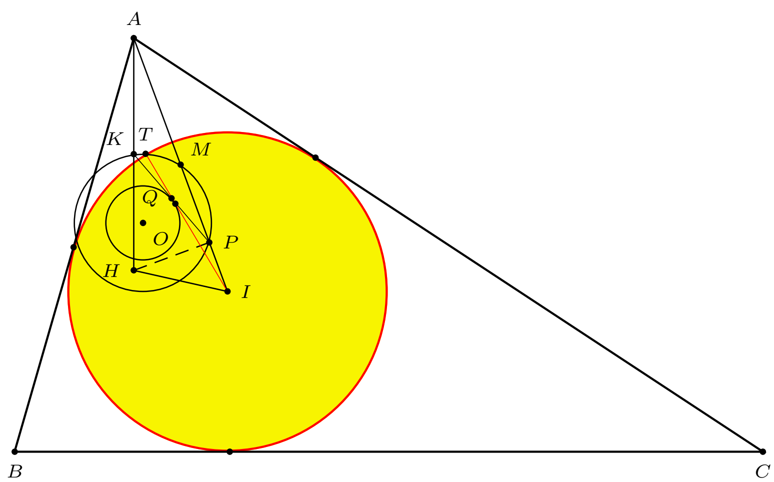

Figure 35

Remark. The construction of a triangle from its circumcircle, incenter, orthocenter was studied by Leonhard Euler [8], who reduced it to the problem of trisection of an angle. In Euler's time, the impossibility of angle trisection by ruler and compass was not yet confirmed.

References

[1] G. L. Alexanderson, A conversation with Leon Bankoff, *College Math. Journal*, 23 (1992) 98–117.
[2] J.-L. Ayme, Sawayama and Thébault's theorem, *Forum Geom.*, 3 (2003) 225–229.
[3] L. Bankoff, Are the twin circles of Archimedes really twin ?, *Math. Mag.* 47 (1974) 214–218.
[4] L. Bankoff, A mixtilinear adventure, *Crux Math.*, 9 (1983) 2–7.
[5] M. Berger, *Geometry II*, Springer-Verlag, 1987.
[6] J. C. Cherry and H. Eves, Problem 1054, *Math. Mag.*, 51 (1978) 305; solution, 53 (1980) 52–53.
[7] H. Dörrie, *100 Great Problems of Elementary Mathematics*, Dover, 1965.
[8] L. Euler, Variae demonstrationes geometriae, *Nova commentarii academiae scientiarum Petropolitanae*, 1 (1747/8), 49–66, also in *Opera Ommnia*, serie prima, vol.26, 15–32.
[9] H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems*, Charles Babbage Research Centre, Winnipeg, 1989.
[10] H. Fukagawa and J. F. Rigby, *Traditional Japanese Mathematics Problems of the 18th and 19th Centuries*, SCT Press, Singapore, 2002.
[11] L. D. Goldstone, V. Thébault and R. Woods, Problem E 1375, *Amer. Math. Monthly*, 66 (1959) 513; solution, 67 (1960) 185–186.
[12] T. L. Heath, *The Works of Archimedes*, 1912, Dover reprint.
[13] D. Klanderman, Teaching and learning mathematics: the influence of constructivism, Chapter 12 of R. W. Howell and W. J. Bradley (ed.), *Mathematics in a Postmodern Age, A Christian Perspective*, pp.333–359, Wm. B Eerdmans, Grand Rapids, Michigan, 2001.
[14] L. Lopes, *Manuel de Construction de Triangles*, QED Texte, Québec, 1996.
[15] J. E. McClure, Start where they are: geometry as an introduction to proof, *Amer. Math. Monthly*, 107 (2000) 44–52.
[16] Report of the M. A. Committee on Geometry, *Math. Gazette*, 2 (1902) 167–172; reprinted in C. Pritchard (ed.) *The Changing Shape of Geometry, Celebrating a Century of Geometry and Geometry Teaching*, 529–536, Cambridge University Press, 2003.
[17] D. E. Smith, *The Teaching of Geometry*, 1911.
[18] V. Thébault, Problem 3887, *Amer. Math. Monthly*, 45 (1938) 482–483.
[19] Wang Yǒngjiàn, Shìtán píngmiàn jǐhé jiàoxué de zuōyòng yǔ dìweì, (On the function and role of the teaching of plane geometry), *Shuxue Tongbao*, 2004, Number 9, 23–24.
[20] W. Wernick, Triangle constructions with three located points, *Math. Mag.*, 55 (1982) 227–230.
[21] P. Woo, Simple constructions of the incircle of an arbelos, *Forum Geom.*, 1 (2001) 133–136.
[22] P. Yiu, Para-Euclidean teaching of Euclidean geometry, in M. K. Siu (ed.) *Restrospect and Outlook on Mathematics Education in Hong Kong, On the Occasion of the Retirement of Dr. Leung Kam Tim*, pp. 215–221, Hong Kong University Press, Hong Kong, 1995.
[23] P. Yiu, Mixtilinear incircles, *Amer. Math. Monthly*, 106 (1999) 952–955.
[24] Zheng yùxìn, Jiàngòu zhǔyì zhī shènsī (Careful consideration of constructivism), *Shuxue Tongbao*, 2004, Number 9, 18–22.

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