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Circles and Triangle Centers Associated with the Lucas Circles

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Abstract. The Lucas circles of a triangle are the three circles mutually tangent to each other externally, and each tangent internally to the circumcircle of the triangle at a vertex. In this paper we present some further interesting circles and triangle centers associated with the Lucas circles.

1. Introduction

In this paper we study circles and triangle centers associated with the three Lucas circles of a triangle. The Lucas circles of a triangle are the three circles mutually tangent to each other externally, and each tangent internally to the circumcircle of the triangle at a vertex.

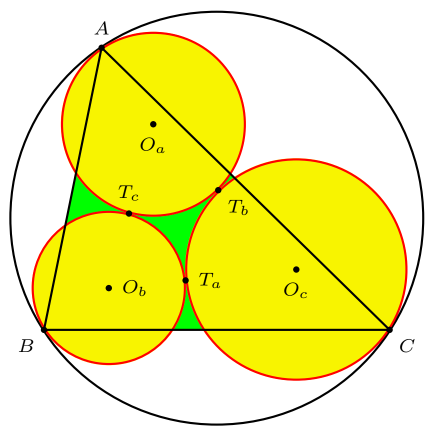

Figure 1

We work with homogeneous barycentric coordinates and make use of John H. Conway's notation in triangle geometry. The indexing of triangle centers follows Kimberling's *Encyclopedia of Triangle Centers* [2]. Many of the triangle centers in this paper are related to the Kiepert perspectors. We recall that given a triangle ABC, the Kiepert perspector $K(\theta)$ is the perspector of the triangle formed by the apices of similar isosceles triangles with base angles θ on the sides of ABC.

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In barycentric coordinates,

$$K(\theta) = \left(\frac{1}{S_A + S_\theta} : \frac{1}{S_B + S_\theta} : \frac{1}{S_C + S_\theta}\right).$$

Its isogonal conjugate is the point

$$K^*(\theta) = (a^2(S_A + S_\theta) : b^2(S_B + S_\theta) : c^2(S_C + S_\theta))$$

on the Brocard axis joining the circumcenter O and the symmedian point K.

2. The centers and points of tangency of the Lucas circles

The Lucas circles $\mathcal{C}_A$, $\mathcal{C}_B$, $\mathcal{C}_C$ of triangle ABC are the images of the circumcircle under the homotheties with centers A, B, C, and ratios $\frac{S}{a^2+S}$, $\frac{S}{b^2+S}$, $\frac{S}{c^2+S}$ respectively. As such they have centers

$$\begin{aligned} O_a =& (a^2(S_A + 2S) : b^2 S_B : c^2 S_C), \\ O_b =& (a^2 S_A : b^2(S_B + 2S) : c^2 S_C), \\ O_c =& (a^2 S_A : b^2 S_B : c^2(S_C + 2S)), \end{aligned}$$

and equations

$$\mathcal{C}_A : \qquad a^2yz + b^2zx + c^2xy - \frac{a^2b^2c^2}{a^2 + S} \cdot (x + y + z)\left(\frac{y}{b^2} + \frac{z}{c^2}\right) = 0,$$

$$\mathcal{C}_B : \qquad a^2yz + b^2zx + c^2xy - \frac{a^2b^2c^2}{b^2 + S} \cdot (x + y + z)\left(\frac{z}{c^2} + \frac{x}{a^2}\right) = 0,$$

$$\mathcal{C}_C : \qquad a^2yz + b^2zx + c^2xy - \frac{a^2b^2c^2}{c^2 + S} \cdot (x + y + z)\left(\frac{x}{a^2} + \frac{y}{b^2}\right) = 0.$$

The Lucas circles are mutually tangent to each other, externally, at

$$\begin{aligned} T_a = \mathcal{C}_B \cap \mathcal{C}_C =& (a^2 S_A : b^2(S_B + S) : c^2(S_C + S)), \\ T_b = \mathcal{C}_C \cap \mathcal{C}_A =& (a^2(S_A + S) : b^2 S_B : c^2(S_C + S)), \\ T_c = \mathcal{C}_A \cap \mathcal{C}_B =& (a^2(S_A + S) : b^2(S_B + S) : c^2 S_C). \end{aligned}$$

See Figure 1. These points of tangency form a triangle perspective with ABC at

$$K^*(\frac{\pi}{4}) = (a^2(S_A + S) : b^2(S_B + S) : c^2(S_C + C)),$$

which is X_{371} of [2].

By Desargues' theorem, the triangles $O_aO_bO_c$ and $T_aT_bT_c$ are perspective. Their perspector is clearly the Gergonne point of triangle $O_aO_bO_c$; it has coordinates

$$(a^2(3S_A + 2S) : b^2(3S_B + 2S) : c^2(3S_C + 2S)).$$

This is the point $K^*(\arctan \frac{3}{2})$.

The exsimilicenter (external center of similitude) of $\mathcal{C}_B$ and $\mathcal{C}_C$ is the point $(0 : b^2 : -c^2)$. Likewise, those of the pairs $\mathcal{C}_C$, $\mathcal{C}_A$ and $\mathcal{C}_A$, $\mathcal{C}_B$ are $(-a^2 : 0 : c^2)$ and $(a^2 - b^2 : 0)$. These three exsimilicenters all lie on the Lemoine axis,

$$\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2} = 0.$$

Proposition 1. *The pedals of O_a on BC, O_b on CA, and O_c on AB form the cevian triangle of the Kiepert perspector $K(\arctan 2)$.*[1]

Proof. These pedals are the points $(0 : 2S_C + S : 2S_B + S)$, $(2S_C + S : 0 : 2S_A + S)$, and $(2S_B + S : 2S_A + S : 0)$. □

Proposition 2. *The pedals of T_a on BC, T_b on CA, and T_c on AB form the cevian triangle of the point $(a^2 + S : b^2 + S : c^2 + S)$.*

Proof. These pedals are the points $(0 : b^2 + S : c^2 + S)$, $(a^2 + S : 0 : c^2)$, and $(a^2 + S : b^2 + S : 0)$. □

3. The radical circle of the Lucas circles

From the equations of the Lucas circles, the radical center of these circles is the point $(x : y : z)$ satisfying

$$\frac{\frac{y}{b^2} + \frac{z}{c^2}}{a^2 + S} = \frac{\frac{z}{c^2} + \frac{x}{a^2}}{b^2 + S} = \frac{\frac{x}{a^2} + \frac{y}{b^2}}{c^2 + S}.$$

This means that $\left(\frac{x}{a^2} : \frac{y}{b^2} : \frac{z}{c^2}\right)$ is the anticomplement of $(a^2 + S : b^2 + S : c^2 + S)$, namely, $(2S_A + S : 2S_B + S : 2S_C + S)$, and the radical center is the point

$$K^*(\arctan 2) = (a^2(2S_A + S) : b^2(2S_B + S) : c^2(2S_C + S)) = X_{1151}$$

on the Brocard axis. Since the Lucas circles are tangent to each other, their radical circle is simply the circle through the tangent points T_a, T_b and T_c. It is also the incircle of triangle $O_aO_bO_c$. As such, it has radius $\frac{2S}{a^2+b^2+c^2+4S} \cdot R$, where R is the circumradius of triangle ABC. Its equation is

$$a^2yz + b^2zx + c^2xy - \frac{2a^2b^2c^2(x + y + z)}{a^2 + b^2 + c^2 + 4S}\left(\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2}\right).$$

4. The inner Soddy circle of the Lucas circles

There are two nonintersecting circles which are tangent to all three Lucas circles. These are the outer and inner Soddy circles of triangle $O_aO_bO_c$. Since the outer Soddy circle is the circumcircle of ABC, the inner Soddy circle is the inverse of this circumcircle with respect to the radical circle. Indeed the points of tangency are the inverses of A, B, C in the radical circle. They are simply the second

[1]This is X_{1131} of [2].

intersections of the lines AT with $\mathcal{C}_a$, BT with $\mathcal{C}_b$, and CT with $\mathcal{C}_c$, where $T = K^*(\arctan 2)$. These are the points

$$(a^2(4S_A+3S) : 2b^2(2S_B+S) : 2c^2(2S_C+S)),$$
$$(2a^2(2S_A+S) : b^2(4S_B+3S) : 2c^2(2S_C+S)),$$
$$(2a^2(2S_A+S) : 2b^2(2S_B+S) : c^2(4S_C+3S)).$$

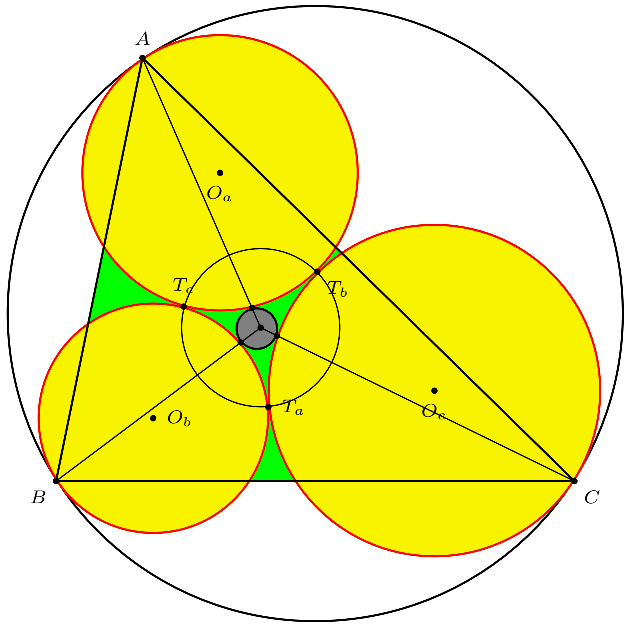

Figure 2

The circle through these points has center $K^*\left(\arctan \frac{7}{4}\right)$ and radius $\frac{2S\cdot R}{4(a^2+b^2+c^2)+14S}$. It has equation

$$a^2yz+b^2zx+c^2xy-\frac{4a^2b^2c^2(x+y+z)}{2(a^2+b^2+c^2)+7S}\left(\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}\right)=0.$$

Proposition 3. *The circumcircle, the radical circle, the inner Soddy circle, and the Brocard circles are coaxal, with the Lemoine axis as radical axis.*

The Brocard circle has equation

$$a^2yz+b^2zx+c^2xy-\frac{a^2b^2c^2(x+y+z)}{a^2+b^2+c^2}\left(\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}\right)=0.$$

The radical trace of these circles, namely, the intersection of the radical axis and the line of centers, is the point

$$(a^2(b^2+c^2-2a^2) : \cdots : \cdots) = K^*(-\arctan(\frac{6S}{a^2+b^2+c^2})).$$

This is X_{187}, the inverse of K in the circumcircle.

5. The Schoute coaxal system

According to [5], the coaxal system of circles containing the circles in Proposition 3 is called the Schoute coaxal system. It has the two isodynamic points as limit points. Indeed, the circle with center X_{187} passing through the isodynamic point $X_{15} = K^*(\frac{\pi}{3})$ is the radical circle of these circles.

Proposition 4. *The circles of the Schoute coaxal system have centers* $K^*(\theta)$ *where* $|\theta| \geq \frac{\pi}{3}$*, and radius* $\left|\frac{\sqrt{\tan^2\theta - 3S}}{2(S_\omega + S\cdot\tan\theta)}\right| \cdot R$*. It has equation*

$$\mathcal{C}_{\rm s}(\theta): \qquad a^2yz + b^2zx + c^2xy - \frac{a^2b^2c^2(x+y+z)}{S_\omega + S\cdot\tan\theta}\left(\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2}\right) = 0.$$

Therefore, a circle with center $(a^2(pS_A + qS) : b^2(pS_B + qS) : c^2(pS_C + qS))$ and square radius $\frac{(p^2-3q^2)a^2b^2c^2}{(2pS+q(a^2+b^2+c^2))^2}$ is the circle $\mathcal{C}_{\rm s}(\arctan\frac{p}{q})$.

circle	$\mathcal{C}_{\rm s}(\theta)$with $\tan\theta =$
circumcircle	∞
Brocard circle	$\cot\omega$
Lemoine axis	$-\cot\omega$
radical circle of Lucas circles	2
inner Soddy circle of Lucas circles	$\frac{7}{4}$

$\theta = \frac{\pi}{3}$ yields the limit point X_{15}.

Proposition 5. *The inversive image of* $\mathcal{C}_{\rm s}(\theta)$ *in* $\mathcal{C}_{\rm s}(\varphi)$ *is the circle* $\mathcal{C}_{\rm s}(\psi)$*, where*

$$\tan\psi = \frac{\tan\theta(\tan^2\varphi + 3) - 6\tan\varphi}{2\tan\theta\tan\varphi - (\tan^2\varphi + 3)}.$$

Corollary 6. (a) *The inverse of* $\mathcal{C}_{\rm s}(\theta)$ *in the circumcircle is* $\mathcal{C}_{\rm s}(-\theta)$.

(b) *The inverse of the circumcircle in* $\mathcal{C}_{\rm s}(\varphi)$ *is the circle* $\mathcal{C}_{\rm s}\left(\arctan\frac{\tan^2\varphi+3}{2\tan\varphi}\right)$.

6. Three infinite families of circles

Let $A'B'C'$ be the circumcevian triangle of the symmedian point K, and $K' = K^*(\frac{\pi}{4})$. The line OA' intersects O_aK' at

$$O_1^a = (a^2(S_A - 2S) : b^2(S_B + 4S) : c^2(S_C + 4S)).$$

This is the center of the circle tangent to the B- and C-Lucas circles, and the circumcircle. It touches the circumcircle at K_0^a. We label this circle $\mathcal{C}_1^a$. The points of tangency with the B- and C-Lucas circles are

$$(a^2(S_A - S) : b^2(S_B + 3S) : c^2(S_C + 2S)),$$
$$(a^2(S_A - S) : b^2(S_B + 2S) : c^2(S_C + 3S))$$

respectively.

Similarly, there are circles $\mathcal{C}_1^b$ and $\mathcal{C}_1^c$ each tangent internally to the circumcircle and externally to two Lucas circles. The centers of the three circles $\mathcal{C}_1^a$, $\mathcal{C}_1^b$, $\mathcal{C}_1^c$ are perspective with ABC at $K^*(\arctan\frac{1}{4})$.

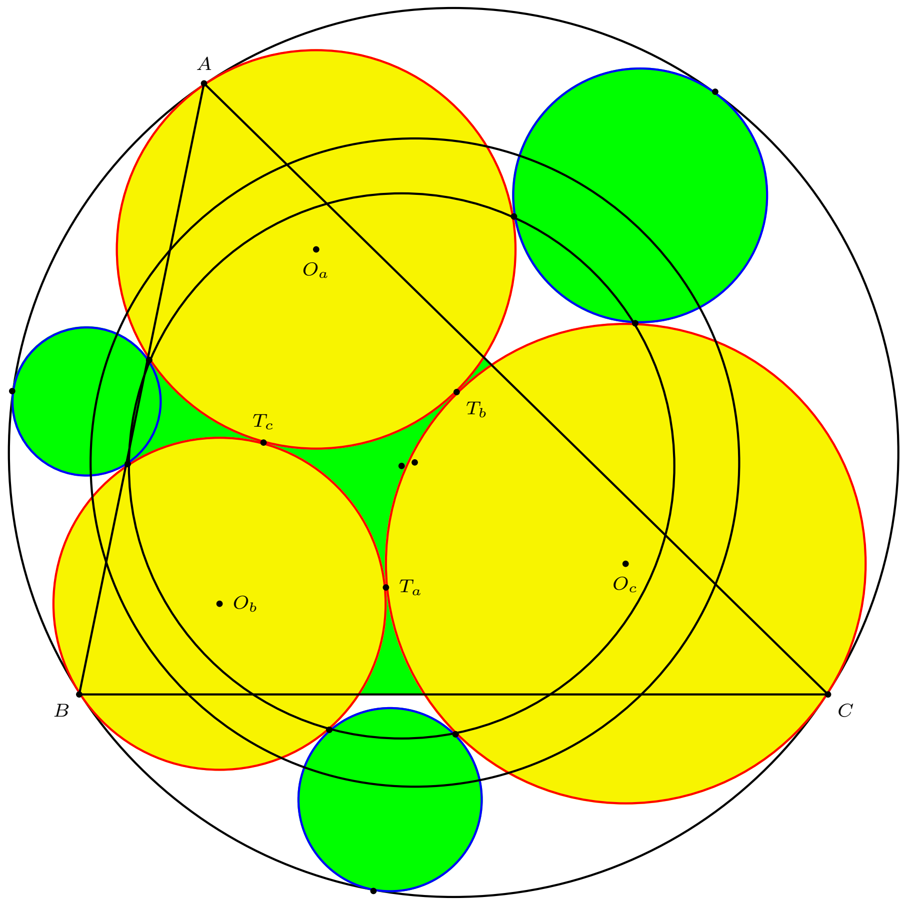

Figure 3

Remarks. (1) The 6 points of tangency with the Lucas circles lie on $\mathcal{C}_s(\arctan 4)$.
(2) The radical circle of these circles is $\mathcal{C}_s(\arctan 6)$. See Figure 3.

The Lucas circles lend themselves to the creation of more and more circle tangencies. There is, for example, an infinite sequence of circles $\mathcal{C}_n^a$ each tangent externally to the B- and C-Lucas circles, so that $\mathcal{C}_n^a$ touches $\mathcal{C}_{n-1}^a$ externally at T_n^a. (We treat $\mathcal{C}_0^a$ as the circumcircle of ABC so that $T_1^a = A'$.

$$O_n^a = (a^2((2n^2-1)S_A - 2nS) : b^2((2n^2-1)S_B + 2n(n+1)S) : c^2((2n^2-1)S_C + 2n(n+1)S)),$$
$$T_n^a = (a^2(2n(n-1)S_A - (2n-1)S) : 2nb^2((n-1)S_B + nS) : 2nc^2((n-1)S_C + nS)).$$

The centers O_n^a of these circles lie on the hyperbola through O_a with foci O_b and O_c. It also contains O and T_a. This is the inner Soddy hyperbola of triangle $O_aO_bO_c$. The points of tangency T_n^a lie on the A-Apollonian circle.

Similarly, we have two analogous families of circles $\mathcal{C}_n^b$ and $\mathcal{C}_n^c$, respectively with centers O_n^b, O_n^c and points of tangency T_n^b, T_n^c.

Remarks. (1) The centers of $\mathcal{C}_n^a$, $\mathcal{C}_n^b$, $\mathcal{C}_n^c$ lie on the circle $\mathcal{C}_s\left(\arctan\frac{4n^2-2n+1}{2n(n-1)}\right)$.

(2) The six points of tangency with the Lucas circles lie on the circle $\mathcal{C}_s\left(\arctan\frac{2n^2+n+1}{n^2}\right)$.

(3) The radical circle of $\mathcal{C}_n^a$, $\mathcal{C}_n^b$, $\mathcal{C}_n^c$ is the circle $\mathcal{C}_s\left(\arctan\frac{2n(2n+1)}{2n^2-1}\right)$.

Proposition 7. *The following pairs of triangles are perspective. The perspectors are all on the Brocard axis.*

Triangle	Triangle	Perspector $= K^*(\theta)$ with $\tan\theta =$
$O_n^aO_n^bO_n^c$	ABC	$\frac{2n^2-1}{2n(n+1)}$
$O_n^aO_n^bO_n^c$	$O_aO_bO_c$	$\frac{3n-1}{2n}$
$O_n^aO_n^bO_n^c$	$T_aT_bT_c$	$\frac{4n+1}{2n}$
$O_n^aO_n^bO_n^c$	circumcevian triangle of K	$\frac{6n^2-3}{2n(n-1)}$
$O_n^aO_n^bO_n^c$	$O_1^aO_1^bO_1^c$	$\frac{5n+3}{2n}$
$O_n^aO_n^bO_n^c$	$O_{n+1}^aO_{n+1}^bO_{n+1}^c$	$\frac{4n^2+6n+3}{2n(n+1)}$
$O_n^aO_n^bO_n^c$	$O_m^aO_m^bO_m^c$	$\frac{4mn+m+n+2}{2mn}$
$T_n^aT_n^bT_n^c$	ABC	$\frac{n-1}{n}$
$T_n^aT_n^bT_n^c$	$O_aO_bO_c$	$\frac{6n^2-2n-1}{4n^2}$
$T_n^aT_n^bT_n^c$	$T_aT_bT_c$	$\frac{4n-1}{2n-1}$
$T_n^aT_n^bT_n^c$	$T_m^aT_m^bT_m^c$	$\frac{4mn-m-n+1}{2mn-m-n}$

7. Centers of similitude

Since the Lucas radical circle, the inner Soddy circle and the circumcircle all belong to the Schoute family, their centers of similitude are all on the Brocard axis.

		Internal	External
inner Soddy circle	circumcircle	$K^*(\arctan 2)$	$K^*(\arctan\frac{3}{2})$
inner Soddy circle	radical circle	$K^*(\arctan\frac{9}{5})$	$K^*(\arctan\frac{5}{3})$

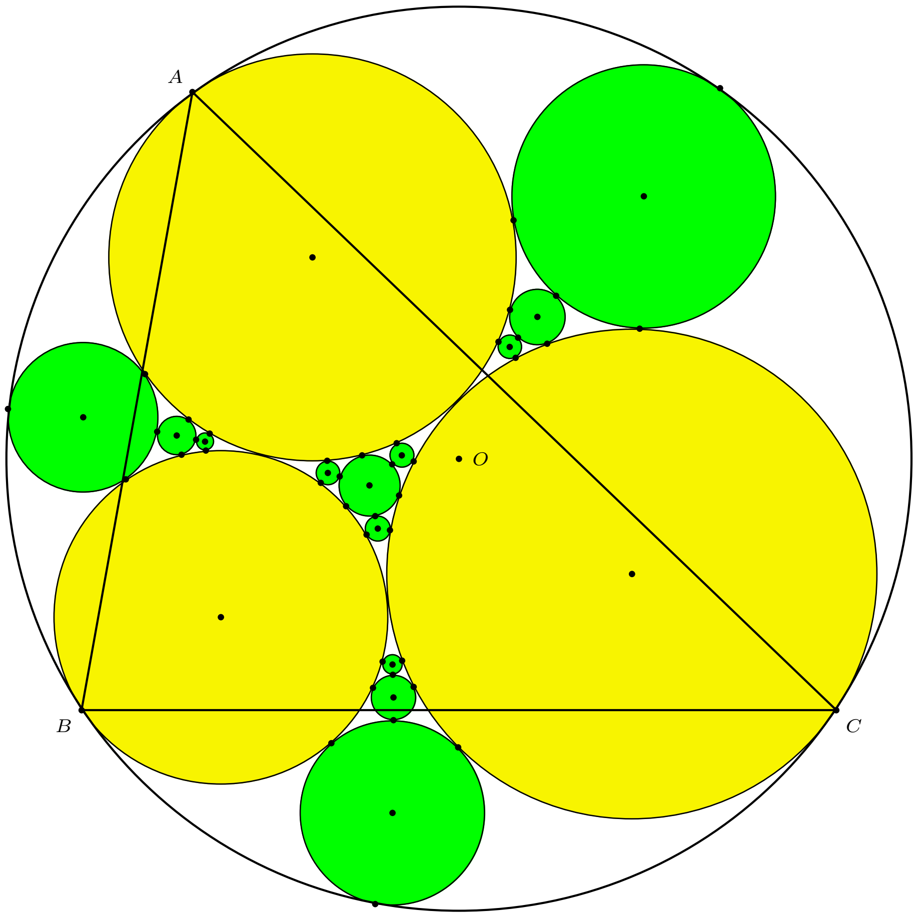

Figure 4

Proposition 8. (a) *The insimilicenters of the Lucas radical circle and the individual Lucas circles form a triangle perspective with* ABC *at* $K^*(\arctan 3)$.

(b) *The exsimilicenters of the Lucas radical circle and the individual Lucas circles form a triangle perspective with* ABC *at* $K^*(\frac{\pi}{4})$.

Proof. These insimilicenters are the points

$$\begin{array}{ccccc}(3a^2(S_A+S) & : & b^2(3S_B+S) & : & c^2(3S_C+S)),\\ (a^2(3S_A+S) & : & 3b^2(S_B+S) & : & c^2(3S_C+S)),\\ (a^2(3S_A+S) & : & b^2(3S_B+S) & : & 3c^2(S_C+S)).\end{array}$$

Likewise, the exsimilicenters are the points

$$\begin{array}{ccccc}(a^2(S_A-S) & : & b^2(S_B+S) & : & c^2(S_C+S)),\\ (a^2(S_A+S) & : & b^2(S_B-S) & : & c^2(S_C+S)),\\ (a^2(S_A+S) & : & b^2(S_B+S) & : & c^2(S_C-S)).\end{array}$$

□

8. Two conics

As explained in [1], the Lucas circles of a triangle are also associated with the inscribed squares of the triangle. We present two interesting conics associated with these inscribed squares. Given a triangle ABC, the A-inscribed square $X_1X_2X_3X_4$ has vertices

$$X_1 = (0 : S_C + S : S_B), \quad \text{and} \quad X_2 = (0 : S_C : S_B + S)$$

on the line BC and

$$X_3 = (a^2 : 0 : S) \quad \text{and} \quad X_4 = (a^2 : S : 0)$$

on AC and AB respectively. It has center $(a^2 : S_C + S : S_B + S)$. Similarly, the coordinates of the $B-$ and C-inscribed squares, and their centers, can be easily written down. It is clear that the centers of these squares form a triangle perspective with ABC at the Kiepert perspector

$$K(\frac{\pi}{4}) = \left(\frac{1}{S_A + S} : \frac{1}{S_B + S} : \frac{1}{S_C + S}\right).$$

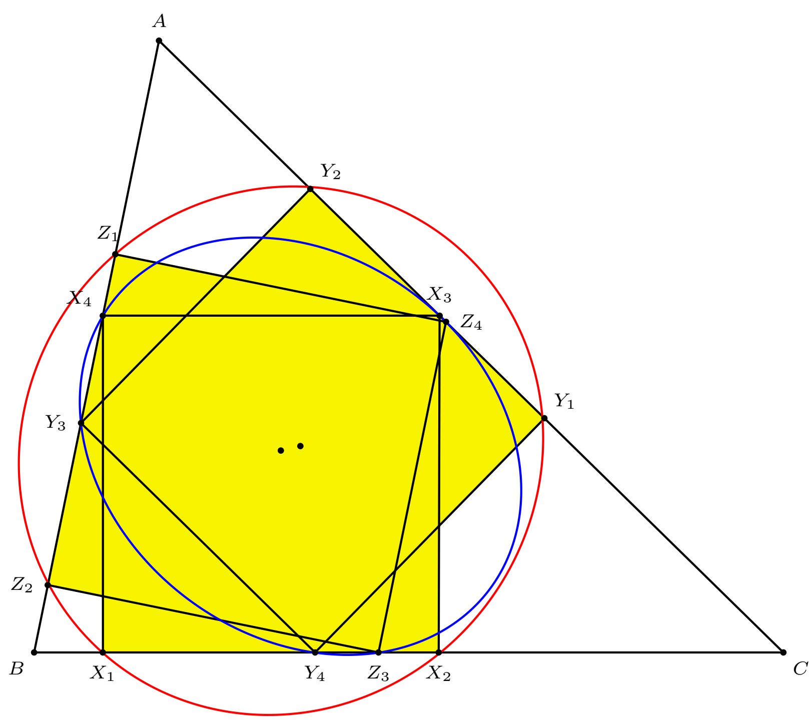

Figure 5.

Proposition 9. *The six points* X_1, X_2, Y_1, Y_2, Z_1, Z_2 *lie on the conic*

$$\sum_{\text{cyclic}}(a^2+S)^2yz = (x+y+z)\sum_{\text{cyclic}} S_A(S_A+S)x.$$

This conic has center $(a^2+S : b^2+S : c^2+S)$.

Proposition 10. *The six points* X_3, X_4, Y_3, Y_4, Z_3, Z_4 *lie on the conic*

$$\sum_{\text{cyclic}}\frac{a^2}{a^2+S}yz = \frac{a^2b^2c^2S(x+y+z)}{(a^2+S)(b^2+S)(c^2+S)}\left(\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}\right).$$

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