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On the Geometry of Equilateral Triangles

József Sándor

Dedicated to the memory of Angela Vasiu (1941-2005)

Abstract. By studying the distances of a point to the sides, respectively the vertices of an equilateral triangle, certain new identities and inequalities are deduced. Some inequalities for the elements of the Pompeiu triangle are also established.

1. Introduction

The equilateral (or regular) triangle has some special properties, generally not valid in an arbitrary triangle. Such surprising properties have been studied by many famous mathematicians, including Viviani, Gergonne, Leibnitz, Van Schooten, Toricelli, Pompeiu, Goormaghtigh, Morley, etc. ([2], [3], [4], [7]). Our aim in this paper is the study of certain identities and inequalities involving the distances of a point to the sides or the vertices of an equilateral triangle. For the sake of completeness, we shall recall some well-known results.

1.1. Let ABC be an equilateral triangle of side length $AB = BC = CA = l$, and height h. Let P be any point in the plane of the triangle. If O is the center of the triangle, then the Leibnitz relation (valid in fact for any triangle) implies that

$$\sum PA^2 = 3PO^2 + \sum OA^2. \tag{1}$$

Let $PO = d$ in what follows. Since in our case $OA = OB = OC = R = \dfrac{l\sqrt{3}}{3}$, we have $\sum OA^2 = l^2$, and (1) gives

$$\sum PA^2 = 3d^2 + l^2. \tag{2}$$

Therefore, $\sum PA^2 =$ constant if and only if $d =$ constant, *i.e.*, when P is on a circle with center O. For a proof by L. Moser via analytical geometry, see [12]. For a proof using Stewart's theorem, see [13].

1.2. Now, let P be in the interior of triangle ABC, and denote by p_a, p_b, p_c its distances from the sides. Viviani's theorem says that

$$\sum p_a = p_a + p_b + p_c = h = \frac{l\sqrt{3}}{2}.$$

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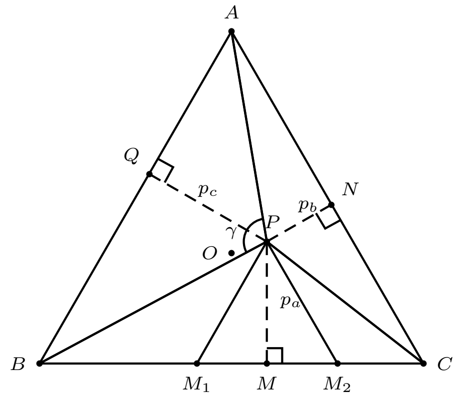

Figure 1

This follows by area considerations, since

$$S(BPC) + S(CPA) + S(APB) = S(ABC),$$

where S denotes area. Thus,

$$\sum p_a = \frac{l\sqrt{3}}{2}. \tag{3}$$

1.3. By Gergonne's theorem one has $\sum p_a^2 =$ constant, when P is on the circle of center O. For such related constants, see for example [13]. We shall obtain more general relations, by expressing $\sum p_a^2$ in terms of l and $d = OP$.

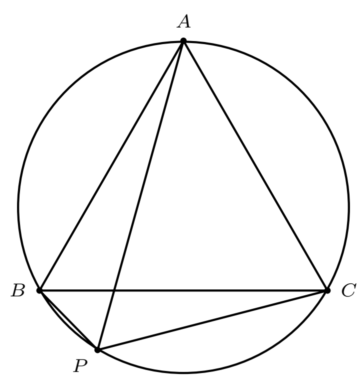

Figure 2

1.4. Another famous theorem, attributed to Pompeiu, states that for any point P in the plane of an equilateral triangle ABC, the distances PA, PB, PC can be the sides of a triangle ([9]-[10], [7], [12], [6]). (See also [1], [4], [11], [15], [16], where extensions of this theorem are considered, too.) This triangle is degenerate if P is on the circle circumscribed to ABC, since if for example P is on the interior or arc BC, then by Van Schooten's theorem,

$$PA = PA + PC. \tag{4}$$

Indeed, by Ptolemy's theorem on $ABPC$ one can write

$$PA \cdot BC = PC \cdot AB + PB \cdot AC,$$

so that $BC = AB = AC = l$ implies (4). For any other positions of P (*i.e.*, P **not** on this circle), by Ptolemy's inequality in quadrilaterals one obtains

$$PA < PB + PC, \quad PB < PA + PC, \text{ and } PC < PA + PB,$$

so that PA, PB, PC are the sides of a triangle. See [13] for many proofs. We shall call a triangle with sides PA, PB, PC a **Pompeiu triangle**. When P is in the interior, the Pompeiu triangle can be explicitly constructed. Indeed, by rotating the triangle ABP with center A through an angle of $60°$, one obtains a triangle $AB'C$ which is congruent to ABP. Then, since $AP = AB' = PB'$, $BP = CB'$, the Pompeiu triangle will be PCB'. Such a rotation will enable us also to compute the area of the Pompeiu triangle.

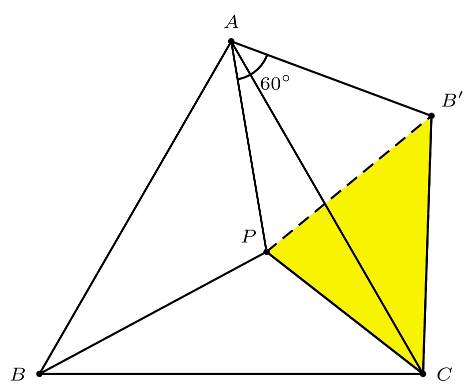

Figure 3

1.5. There exist many known inequalities for the distances of a point to the vertices of a triangle. For example, for any point P and any triangle ABC,

$$\sum PA \geq 6r, \tag{5}$$

where r is the radius of incircle (due to M. Schreiber (1935), see [7], [13]). Now, in our case $6r = l\sqrt{3}$, (5) gives

$$\sum PA \geq l\sqrt{3} \tag{6}$$

for any point P in the plane of equilateral triangle ABC. For an independent proof see [12, p.52]. This is based on the following idea: let M_1 be the midpoint of BC. By the triangle inequality one has $AP + PM_1 \geq AM_1$. Now, it is well known that

$PM_1 \leq \dfrac{PB+PC}{2}$. From this, we get $l\sqrt{3} \leq 2PA + PB + PC$, and by writing two similar relations, the relation (6) follows after addition. We note that already (2) implies $\sum PA^2 \geq l^2$, but (6) offers an improvement, since

$$\sum PA^2 \geq \frac{1}{3}\left(\sum PA\right)^2 \geq l^2 \tag{7}$$

by the classical inequality $x^2+y^2+z^2 \geq \dfrac{1}{3}(x+y+z)^2$. As in (7), equality holds in (6) when $P \equiv O$.

2. Identities for p_a, p_b, p_c

Our aim in this section is to deduce certain identities for the distances of an interior point to the sides of an equilateral triangle ABC.

Let P be in the interior of triangle ABC (see Figure 1). Let $PM \perp BC$, etc., where $PM = p_a$, etc. Let $PM_1 \| AB$, $PM_2 \| AC$. Then triangle PM_1M_2 is equilateral, giving $\overrightarrow{PM} = \dfrac{\overrightarrow{PM}_1 + \overrightarrow{PM}_2}{2}$. By writing two similar relations for $\overrightarrow{PQ}$ and $\overrightarrow{PN}$, and using $\overrightarrow{PO} = \dfrac{\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}}{3}$, one easily can deduce the following vectorial identity:

$$\overrightarrow{PM} + \overrightarrow{PN} + \overrightarrow{PQ} = \frac{3}{2}\overrightarrow{PO}. \tag{8}$$

Since $\overrightarrow{PM} \cdot \overrightarrow{PN} = PM \cdot PN \cdot \cos 120^\circ = -\dfrac{1}{2} PM \cdot PN$ (in the cyclic quadrilateral $CNPM$), by putting $PO = d$, one can deduce from (8)

$$\sum PM^2 + \frac{1}{2}\sum \overrightarrow{PM} \cdot \overrightarrow{PN} = \frac{9}{4}PO^2,$$

so that

$$\sum p_a^2 - \sum p_a p_b = \frac{9}{4}d^2. \tag{9}$$

For similar vectorial arguments, see [12]. On the other hand, from (3), we get

$$\sum p_a^2 + 2\sum p_a p_b = \frac{3l^2}{4}. \tag{10}$$

Solving the system (9), (10) one can deduce the following result.

Proposition 1.

$$\sum p_a^2 = \frac{l^2+6d^2}{4}, \tag{11}$$

$$\sum p_a p_b = \frac{l^2-3d^2}{4}. \tag{12}$$

There are many consequences of (11) and (12). First, $\sum p_a^2 =$ constant if and only if $d =$ constant, *i.e.*, P lying on a circle with center O. This is Gergonne's theorem. Similarly, (12) gives $\sum p_a \cdot p_b =$ constant if and only if $d =$ constant, *i.e.*, P again lying on a circle with center O. Another consequence of (11) and (12) is

$$\sum p_a p_b \le \frac{l^2}{4} \le \sum p_a^2. \tag{13}$$

An interesting connection between $\sum PA^2$ and $\sum p_a^2$ follows from (2) and (11):

$$\sum PA^2 = 2\sum p_a^2 + \frac{l^2}{2}. \tag{14}$$

3. Inequalities connecting p_a, p_b, p_c with PA, PB, PC

This section contains certain new inequalities for PA, p_a, etc. Among others, relation (18) offers an improvement of known results.

By the arithmetic-geometric mean inequality and (3), one has

$$p_a p_b p_c \le \left(\frac{p_a + p_b + p_c}{3}\right)^3 = \left(\frac{l\sqrt{3}}{6}\right)^3 = \frac{l^3\sqrt{3}}{72}.$$

Thus,

$$p_a p_b p_c \le \frac{l^3\sqrt{3}}{72} \tag{15}$$

for any interior point P of equilateral triangle ABC. This is an equality if and only only if $p_a = p_b = p_c$, *i.e.*, $P \equiv O$.

Now, let us denote $\alpha = mes(\sphericalangle BPC)$, etc. Writing the area of triangle BPC in two ways, we obtain

$$BP \cdot CP \cdot \sin\alpha = l \cdot p_a.$$

Similarly,

$$AP \cdot BP \cdot \sin\gamma = l \cdot p_c, \quad AP \cdot CP \cdot \sin\beta = l \cdot p_c.$$

By multiplying these three relations, we have

$$PA^2 \cdot PB^2 \cdot PC^2 = \frac{l^3 p_a p_b p_c}{\sin\alpha \sin\beta \sin\gamma}. \tag{16}$$

We now prove the following result.

Theorem 2. *For an interior point P of an equilateral triangle ABC, one has*

$$\prod PA^2 \ge \frac{8l^3}{3\sqrt{3}} \prod p_a \quad \textit{and} \quad \sum PA \cdot PB \ge l^2.$$

Proof. Let $f(x) = \ln \sin x$, $x \in (0, \pi)$. Since $f''(x) = -\dfrac{1}{\sin^2 x} < 0$, f is concave, and

$$f\left(\frac{\alpha+\beta+\gamma}{3}\right) \geq \frac{f(\alpha)+f(\beta)+f(\gamma)}{3},$$

giving

$$\prod \sin \alpha \leq \frac{3\sqrt{3}}{8}, \tag{17}$$

since $\dfrac{\alpha+\beta+\gamma}{3} = 120^\circ$ and $\sin 120^\circ = \dfrac{\sqrt{3}}{2}$. Thus, (16) implies

$$\prod PA^2 \geq \frac{8l^3}{3\sqrt{3}} \prod p_a. \tag{18}$$

We note that $\dfrac{8l^3}{3\sqrt{3}} \prod p_a \geq 64 \prod p_a^2$, since this is equivalent to $\prod p_a \leq \dfrac{l^3\sqrt{3}}{72}$, i.e. relation (15). Thus (18) improves the inequality

$$\prod PA \geq 8 \prod p_a \tag{19}$$

valid for any triangle (see [2, inequality 12.25], or [12, p.46], where a slightly improvement appears).

On the other hand, since $\dfrac{\alpha}{2} + \dfrac{\beta}{2} + \dfrac{\gamma}{2} = 180^\circ$, one has

$$\begin{aligned}
&\cos\alpha + \cos\beta + \cos\gamma + \frac{3}{2} \\
=&2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2\cos^2\frac{\gamma}{2} + \frac{1}{2} \\
=&2\left(\cos^2\frac{\gamma}{2} - \cos\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2} + \frac{1}{4}\right) \\
=&2\left(\cos^2\frac{\gamma}{2} - \cos\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2} + \frac{1}{4}\cos^2\frac{\alpha-\beta}{2} + \frac{1}{4}\sin^2\frac{\alpha-\beta}{2}\right) \\
=&2\left[\left(\cos\frac{\gamma}{2} - \frac{1}{2}\cos\frac{\alpha-\beta}{2}\right)^2 + \frac{1}{4}\sin^2\frac{\alpha-\beta}{2}\right] \geq 0,
\end{aligned}$$

with equality only for $\alpha = \beta = \gamma = 120^\circ$. Thus:

$$\cos\alpha + \cos\beta + \cos\gamma \geq -\frac{3}{2} \tag{20}$$

for any α, β, γ satisfying $\alpha + \beta + \gamma = 360^\circ$.

Now, in triangle APB one has, by the law of cosines,

$$l^2 = PA^2 + PB^2 - 2PA \cdot PB \cdot \cos\gamma,$$

giving

$$\cos\gamma = \frac{PA^2 + PB^2 - l^2}{2PA \cdot PB}.$$

By writing two similar relations, one gets, by (20),

$$\frac{PA^2+PC^2-l^2}{2PA\cdot PC}+\frac{PB^2+PC^2-l^2}{2PB\cdot PC}+\frac{PA^2+PB^2-l^2}{2PA\cdot PB}+\frac{3}{2}\geq 0,$$

so that

$$\begin{aligned}&(PA^2\cdot PB+PB^2\cdot PA+PA\cdot PB\cdot PC)\\&+(PC^2\cdot PB+PB^2\cdot PC+PA\cdot PB\cdot PC)\\&+(PA^2\cdot PC+PC^2\cdot PA+PA\cdot PB\cdot PC)\\&-l^2(PA+PB+PC)\\&\geq 0.\end{aligned}$$

This can be rearranged as

$$(PA+PB+PC)\left(\sum PA\cdot PB-l^2\right)\geq 0,$$

and gives the inequality

$$\sum PA\cdot PB\geq l^2, \tag{21}$$

with equality when $P\equiv O$. □

4. The Pompeiu triangle

In this section, we deduce many relations connecting PA, PB, PC, etc by obtaining an identity for the area of Pompeiu triangle. In particular, a new proof of (21) will be given.

4.1. Let P be a point inside the equilateral triangle ABC (see Figure 3). The Pompeiu triangle $PB'C$ has the sides PA, PB, PC. Let R be the radius of circumcircle of this triangle. It is well known that $\sum PA^2\leq 9R^2$ (see [1, p.171], [6, p.52], [9, p.56]). By (2) we get

$$R^2\geq\frac{l^2+3d^2}{9}\geq\frac{l^2}{9}, \tag{22}$$

$$R\geq\frac{l}{3}, \tag{23}$$

with equality only for $d=0$, *i.e.*, $P\equiv O$. Inequality (23) can be proved also by the known relation $s\leq\frac{3R\sqrt{3}}{2}$, where s is the semi-perimeter of the triangle. Thus we obtain the following inequalities.

Proposition 3.

$$3R\sqrt{3}\geq\sum PA\geq l\sqrt{3}, \tag{24}$$

where the last inequality follows by (6).

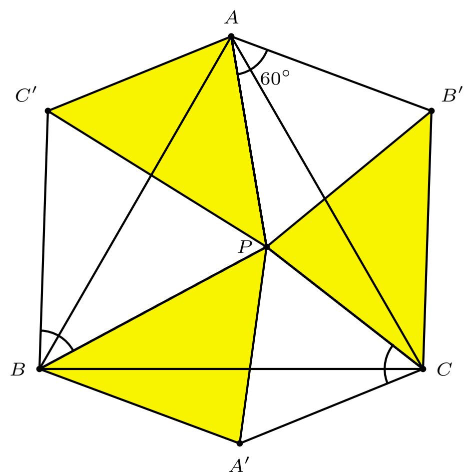

Figure 4

Now, in order to compute the area of the Pompeiu triangle, let us make two similar rotations as in Figure 3, *i.e.*, a rotation of angle 60° with center C of triangle APC, and another with center B of BPC. We shall obtain a hexagon (see Figure 4), $AB'CA'BC'$, where the Pompeiu triangles PBA', PAC', PCB' have equal area T. Since $\triangle APC \equiv \triangle BA'C$, $\triangle APB \equiv \triangle AB'C$, $\triangle AC'B \equiv \triangle BPC$, the area of hexagon $= 2\text{Area}(ABC)$. But $\text{Area}(APB') = \dfrac{AP^2\sqrt{3}}{4}$, APB' being an equilateral triangle. Therefore,

$$\frac{2l^2\sqrt{3}}{4} = 3T + \frac{PA^2\sqrt{3}}{4} + \frac{PB^2\sqrt{3}}{4} + \frac{PC^2\sqrt{3}}{4},$$

which by (2) implies

$$T = \frac{\sqrt{3}}{12}(l^2 - 3d^2). \tag{25}$$

Theorem 4. *The area of the Pompeiu triangle is given by relation (25).*

Corollary 5.

$$T \leq \frac{\sqrt{3}}{12}l^2, \tag{26}$$

with equality when $d = 0$, i.e., *when* $P \equiv O$.

Now, since in any triangle of area T, and sides PA, PB, PC one has

$$2\sum PA \cdot PB - \sum PA^2 \geq 4\sqrt{3} \cdot T$$

(see for example [14], relation (8)), by (2) and (25) one can write

$$2\sum PA \cdot PB \geq 3d^2 + l^2 + l^2 - 3d^2 = 2l^2,$$

giving a new proof of (21).

Corollary 6.

$$\sum PA^2 \cdot PB^2 \geq \frac{1}{3}\left(\sum PA \cdot PB\right)^2 \geq \frac{l^4}{3}. \tag{27}$$

4.2. Note that in any triangle, $\sum PA^2 \cdot PB^2 \geq \frac{16}{9}S^2$, where $S = \mathrm{Area}(ABC)$ (see [13, pp.31-32]). In the case of equilateral triangles, (27) offers an improvement.

Since $r = \frac{T}{s}$, where s is the semi-perimeter and r the radius of inscribed circle to the Pompeiu triangle, by (6) and (26) one can write

$$r \leq \frac{\left(\frac{\sqrt{3}}{12}l^2\right)}{\left(\frac{l\sqrt{3}}{2}\right)} = \frac{l}{6}.$$

Thus, we obtain the following result.

Proposition 7. *For the radii r and R of the Pompeiu triangle one has*

$$r \leq \frac{l}{6} \leq \frac{R}{2}. \tag{28}$$

The last inequality holds true by (23). This gives an improvement of Euler's inequality $r \leq \frac{R}{2}$ for the Pompeiu triangle. Since $T = \frac{PA \cdot PB \cdot PC}{4R}$, and $r = \frac{T}{s}$, we get

$$PA \cdot PB \cdot PC = 2Rr(PA + PB + PC),$$

and the following result.

Proposition 8.

$$PA \cdot PB \cdot PC \geq \frac{2l^2 r\sqrt{3}}{3} \geq 4r^2 l\sqrt{3}. \tag{29}$$

The last inequality is the first one of (28). The following result is a counterpart of (29).

Proposition 9.

$$PA \cdot PB \cdot PC \leq \frac{\sqrt{3}l^2 R}{3}. \tag{30}$$

This follows by $T = \frac{PA \cdot PB \cdot PC}{4R}$ and (26).

4.3. The sides PA, PB, PC can be expressed also in terms of p_a, p_b, p_c. Since in triangle PNM (see Figure 1), $\sphericalangle NPM = 120°$, by the Law of cosines one has

$$MN^2 = PM^2 + PN^2 - 2PM \cdot PN \cdot \cos 120°.$$

On the other hand, in triangle NMC, $NM = PC \cdot \sin C$, PC being the diameter of circumscribed circle. Since $\sin C = \sin 60° = \frac{\sqrt{3}}{2}$, we have $MN = PC\frac{\sqrt{3}}{2}$, and the following result.

Proposition 10.

$$PC^2 = \frac{4}{3}(p_b^2 + p_a^2 + p_a p_b). \tag{31}$$

Similarly,

$$PA^2 = \frac{4}{3}(p_b^2 + p_c^2 + p_b p_c), \quad PB^2 = \frac{4}{3}(p_c^2 + p_a^2 + p_c p_a). \tag{32}$$

In theory, all elements of Pompeiu's triangle can be expressed in terms of p_a, p_b, p_c. We note that by (11) and (12) relation (2) can be proved again. By the arithmetic-geometric mean inequality, we have

$$\prod PA^2 \leq \left(\frac{\sum PA^2}{3}\right)^3,$$

and the following result.

Theorem 11.

$$PA \cdot PB \cdot PC \leq \left(\frac{l^2 + 3d^2}{3}\right)^{3/2}. \tag{33}$$

On the other hand, by the Pólya-Szegö inequality in a triangle (see [8], or [14]) one has

$$T \leq \frac{\sqrt{3}}{4}(PA \cdot PB \cdot PC)^{2/3},$$

so by (25) one can write (using (12)):

Theorem 12.

$$PA \cdot PB \cdot PC \geq \left(\frac{l^2 - 3d^2}{3}\right)^{3/2} = \left(\frac{4\sum p_a p_b}{3}\right)^{3/2}. \tag{34}$$

4.4. Other inequalities may be deduced by noting that by (31),

$$(p_a + p_b)^2 \leq PC^2 \leq 2(p_a^2 + p_b^2).$$

Since $(\sqrt{x} + \sqrt{y} + \sqrt{z})^2 \leq 3(x + y + z)$ applied to $x = p_a^2 + p_b^2$, etc., we get

$$\sum PA \leq 4\sqrt{3} \cdot \sqrt{p_a^2 + p_b^2 + p_c^2},$$

i.e. by (11) we deduce the following inequality.

Theorem 13.

$$\sum PA \leq \sqrt{3(l^2 + 6d^2)}. \tag{35}$$

This is related to (6). In fact, (6) and (35) imply that $\sum PA = l\sqrt{3}$ if and only if $d = 0$, *i.e.*, $P \equiv O$.

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József Sándor: Babeş-Bolyai University of Cluj, Romania
E-mail address: jsandor@math.ubbcluj.ro, jjsandor@hotmail.com