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Construction of Brahmagupta n-gons

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Abstract. The Indian mathematician Brahmagupta's contributions to mathematics and astronomy are well known. His principle of adjoining Pythagorean triangles to construct general Heron triangles and cyclic quadrilaterals having integer sides, diagonals and area can be employed to appropriate Heron triangles themselves to construct any inscribable n-gon, $n \geq 3$, that has integer sides, diagonals and area. To do so we need a different description of Heron triangles by families that contain a common angle. In this paper we describe such a construction.

1. Introduction

A right angled triangle with rational sides is called a rational Pythagorean triangle. This has rational area. When these rationals are integers, it is called a Pythagorean triangle. More generally, an n-gon with rational sides, diagonals and area is called a rational Heron n-gon, $n \geq 3$. When these rationals are converted into integers by a suitable similarity transformation we obtain a Heron n-gon. If a Heron n-gon is cyclic, *i.e.*, inscribable in a circle then we obtain a Brahmagupta n-gon. In this journal and elsewhere a number of articles have appeared on various descriptions of Heron triangles and Brahmagupta quadrilaterals. Some of these are mentioned in the references. Hence we assume familiarity with the basic geometric and trigonometric results. Also, the knowledge of Pythagorean triples is assumed.

We may look upon the family of Pythagorean triangles as the particular family of Heron triangles that contain a right angle. This suggests that the complete set of Heron triangles may be described by families that contain a common Heron angle (A Heron angle has its sine and cosine rational). Once this is done we may look upon the Brahmagupta principle as follows: He took two Heron triangles ABC and $A'B'C'$ that have $\cos A + \cos A' = 0$ and ajoined them along a common side to describe Heron triangles. This enables us to generalize the Brahmagupta principle to members of appropriate families of Heron triangles to construct rational Brahmagupta n-gons, $n \geq 3$. A similarity transformation assures that these rationals can be rendered integers to obtain a Brahmagupta n-gon, $n \geq 3$.

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2. Description of Heron triangles by angle families

In the interest of clarity and simplicity we first take a numerical example and then give the general result [4]. Suppose that we desire the description of the family of Heron triangles ABC each member of which contains the common Heron angle given by $\cos A = \frac{3}{5}$. The cosine rule applied to a member of that family shows that the sides (a, b, c) are related by the equation

$$a^2 = b^2 + c^2 - \frac{6}{5}bc = \left(b - \frac{3}{5}c\right)^2 + \left(\frac{4}{5}c\right)^2.$$

Since a, b, c are natural numbers the triple a, $b - \frac{3}{5}c$, $\frac{4}{5}c$ must be a Pythagorean triple. That is to say

$$a = \lambda(u^2 + v^2), \quad b - \frac{3}{5}c = \lambda(u^2 - v^2), \quad \frac{4}{5}c = \lambda(2uv).$$

In the above, u, v are relatively prime natural numbers and $\lambda = 1, 2, 3, \ldots$. The least value of λ that makes c integral is 2. Hence we have the description

$$(a, b, c) = (2(u^2 + v^2),\ (u + 2v)(2u - v),\ 5uv), \quad (u, v) = 1,\ u > \frac{1}{2}v. \quad (1)$$

A similar procedure determines the Heron triangle family $A'B'C'$ that contains the supplementary angle of A, *i.e.*, $\cos A' = -\frac{3}{5}$:

$$(a, b, c) = (2(u^2 + v^2),\ (u - 2v)(2u + v),\ 5uv), \quad (u, v) = 1,\ u > 2v. \quad (2)$$

The reader is invited to check that the family (1) has $\cos A = \frac{3}{5}$ and that (2) has $\cos A' = -\frac{3}{5}$ independently of u and v.

More generally the Heron triangle family determining the common angle A given by $\cos A = \frac{p^2 - q^2}{p^2 + q^2}$ and the supplementary angle family generated by $\cos A' = -\frac{p^2 - q^2}{p^2 + q^2}$ are given respectively by

$$\begin{aligned}(a, b, c) = \ & (pq(u^2 + v^2),\ (pu - qv)(qu + pv),\ (p^2 + q^2)uv), \\ & (u, v) = (p, q) = 1,\ u > \tfrac{q}{p}v \quad \text{and } p > q.\end{aligned} \quad (3)$$

$$\begin{aligned}(a', b', c') = \ & (pq(u^2 + v^2),\ (pu + qv)(qu - pv),\ (p^2 + q^2)uv), \\ & (u, v) = (p, q) = 1,\ u > \tfrac{p}{q}v \quad \text{and } p > q.\end{aligned} \quad (4)$$

Areas of (3) and (4) are given by $\frac{1}{2}bc \sin A$ and $\frac{1}{2}b'c' \sin A'$ respectively. Notice that $p = 2$, $q = 1$ in (3) and (4) yield (1) and (2) and that $\angle BAC$ and $\angle B'A'C'$ are supplementary angles. Hence these triangles themselves can be adjoined when $u > \frac{p}{q}v$. The consequences are better understood by a numerical illustration:

$u = 5$, $v = 1$ in (1) and (2) yield $(a, b, c) = (52, 63, 25)$ and $(a', b', c') = (52, 33, 25)$. These can be adjoined along the common side 25. See Figure 1. The result is the isosceles triangle $(96, 52, 52)$ that reduces to $(24, 13, 13)$. As a matter of fact the families (1) and (2) or (3) and (4) may be adjoined likewise to describe the complete set of isosceles Heron triangles:

$$(a,b,c) = (2(u^2 - v^2),\ u^2 + v^2,\ u^2 + v^2), \quad u > v,\ (u,v) = 1. \tag{5}$$

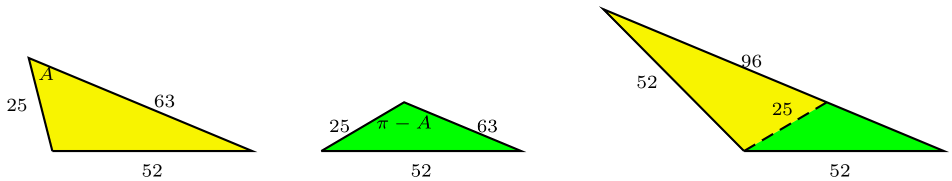

Figure 1

As mentioned in the beginning of this section, the general cases involve routine algebra so the details are left to the reader.

However, the families (1) and (2) or (3) and (4) may be adjoined in another way. This generates the complete set of Heron triangles. Again, we take a numerical illustration.

$u = 3$, $v = 2$ in (1) yields $(a,b,c) = (13,14,15)$ (after reduction by the gcd of (a,b,c)). Now we put different values for u, v in (2), say, $u = 4$, $v = 1$. This yields $(a',b',c') = (17,9,10)$. It should be remembered that we still have $\angle BAC + \angle B'A'C' = \pi$. As they are, triangles ABC and $A'B'C'$ cannot be adjoined. They must be enlarged suitably by similarity transformations to have $AB = A'B'$, and then adjoined. See Figure 2.

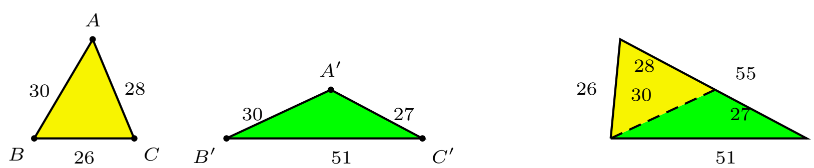

Figure 2

The result is the new Heron triangle $(55,26,51)$. More generally, if we put $u = u_1$, $v = v_1$ in (1) or(3) and $u = u_2$, $v = v_2$ in (2) or (4) and after applying the necessary similiarity transformations, the adjoin (after reduction by the gcd) yields

$$(a,b,c) = (u_1v_1(u_2^2 + v_2^2),\ (u_1^2 - v_1^2)u_2v_2 + (u_2^2 - v_2^2)u_1v_1,\ u_2v_2(u_1^2 + v_1^2). \tag{6}$$

This is the same description of Heron triangles that Euler and others obtained [1]. Now we easily see that Brahmagupta took the case of $p = q$ in (3) and (4).

In the next section we extend this remarkable adjoining idea to generate Brahmagupta n-gons, $n > 3$. At this point recall Ptolemy's theorem on convex cyclic quadrilaterals: *The product of the diagonals is equal to the sum of the products of the two pairs of opposite sides*. Here is an important observation: In a convex cyclic quadrilateral with sides a, b, c, d in order and diagonals e, f, Ptolemy's theorem, viz., $ef = ac + bd$ shows that if five of the preceding elements are rational then the sixth one is also rational.

3. Construction of Brahmagupta n-gons, $n > 3$

It is now clear that we can take any number of triangles, either all from one of the families or some from one family and some from the supplementary angle family and place them appropriately to construct a Brahmagupta n-gon. To convince the reader we do illustrate by numerical examples. We extensively deal with the case $n = 4$. This material is different from what has appeared in [5, 6]. The following table shows the primitive (a, b, c) and the suitably enlarged one, also denoted by (a, b, c). T_1 to T_6 are family (1) triangles, and T_7, T_8 are family (2) triangles. These triangles will be used in the illustrations to come later on.

Table 1: Heron triangles

	u	v	Primitive (a, b, c)	Enlarged (a, b, c)
T_1	3	1	$(4, 5, 3)$	$(340, 425, 255)$
T_2	4	1	$(17, 21, 10)$	$(340, 420, 200)$
T_3	5	3	$(68, 77, 75)$	$(340, 385, 375)$
T_4	7	6	$(85, 76, 105)$	$(340, 304, 420)$
T_5	9	2	$(85, 104, 45)$	$(340, 416, 180)$
T_6	13	1	$(68, 75, 13)$	$(340, 375, 65)$
T_7	4	1	$(17, 9, 10)$	$(340, 180, 200)$
T_8	13	1	$(340, 297, 65)$	$(340, 297, 65)$

The same or different Heron triangles can be adjoined in different ways. We first show this in the illustration of the case of quadrilaterals. Once the construction process is clear, the case of $n > 4$ would be analogous to that $n = 4$. Hence we just give one illustration of $n = 5$ and $n = 6$.

3.1. *Brahmagupta quadrilaterals.* The Brahmagupta quadrilateral can be generated in the following ways:

(i) A triangle taken from family (1) (respectively (3)) or family (2) (respectively (4), henceforth this is to be understood) adjoined with itself,

(ii) two different triangles taken from the same family adjoined,

(iii) one triangle taken from family (1) adjoined with a triangle from family (2).

Here are examples of each case.

Example 1. We take the primitive $(a, b, c) = (17, 21, 10)$, *i.e.*, T_2 and adjoin with itself (see Figure 3). Since $\angle CAD = \angle CBD$, $ABCD$ is cyclic. Ptolemy's theorem shows that $AB = \frac{341}{17}$ is rational. By enlarging the sides and diagonals 17 times each we get the Brahmagupta quadrilateral $ABCD$, in fact a trapezoid, with

$$AB = 341,\ BC = AD = 170,\ CD = 289,\ AC = BD = 357.$$

See Figure 3. Rather than calculating the actual area, we give an argument that shows that the area is integral. This is so general that it is applicable to other adjunctions to follow in our discussion.

Since $\angle BAC = \angle BDC$, $\angle ABD = \angle ACD$, and $\angle BAD = \angle BAC + \angle CAD$, $\angle BAD$ is also a Heron angle and that triangle ABD is Heron. (Note:

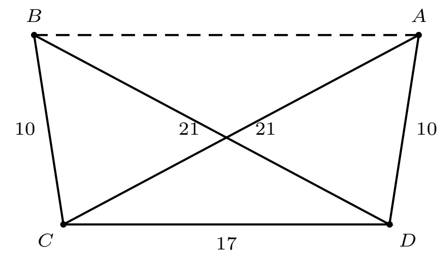

Figure 3

If α and β are Heron angles then $\alpha \pm \beta$ are also Heron angles. To see this consider $\sin(\alpha \pm \beta)$ and $\cos(\alpha \pm \beta)$). $ABCD$ being the disjoint sum of the Heron triangles BCD and BDA, its area must be integral.

This particular adjunction can be done along any side, ***i.e.***, 17, 10, or 21. However, such a liberty is not enjoyed by the remaining constructions which involve adjunction of different Heron triangles. We leave it to the reader to figure out why.

Example 2. We adjoin the primitive triangles T_4, T_5 from Table 1. This can be done in two ways.

(i) Figure 4A illustrates one way. As in Example 1, $AB = \frac{1500}{17}$, so Figure 4A is enlarged 17 times. The area is integral (reasoned as above). Hence the resulting quadrilateral is Brahmagupta.

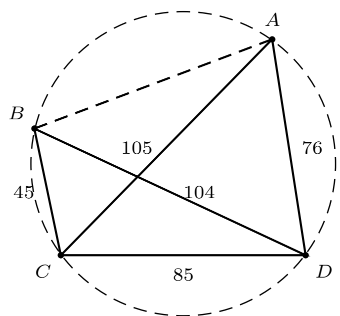

Figure 4A

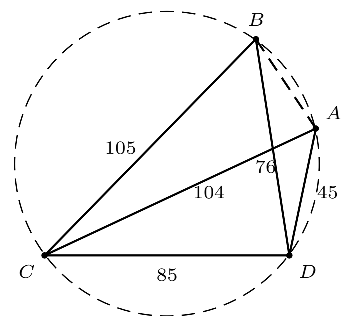

Figure 4B

(ii) Figure 4B illustrates the second adjunction in which the vertices of one base are in reverse order. In this case, $AB = \frac{187}{5}$ hence the figure needs only five times enlargement. Henceforth, we omit the argument to show that the area is integral.

Example 3. We adjoin the primitive triangles T_1 and T_7, which contain supplementary angles A and $\pi - A$. Here, too two ways are possible. In each case no enlargement is necessary. See Figures 5A and 5B.

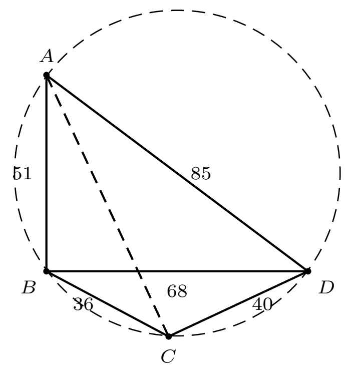

Figure 5A

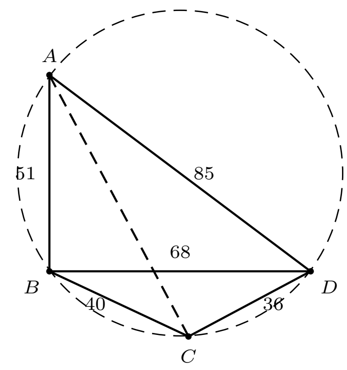

Figure 5B

3.2. *Brahmagupta pentagons.* To construct a Brahmagupta pentagon we need three Heron triangles, in general, taken either all from (1) or some from (1) and the rest from (2) in any combination. Here, too, one triangle can be used twice as in Example 1 above. Hence, a Brahmagupta pentagon can be constructed in more than two ways. We give just one illustration using the (enlarged) triangles T_3, T_4, and T_7. The reader is invited to play the adjuction game using these to consider all possibilities, *i.e.*, T_3, T_3, T_4; T_3, T_4, T_4; T_7, T_7, T_3 etc.

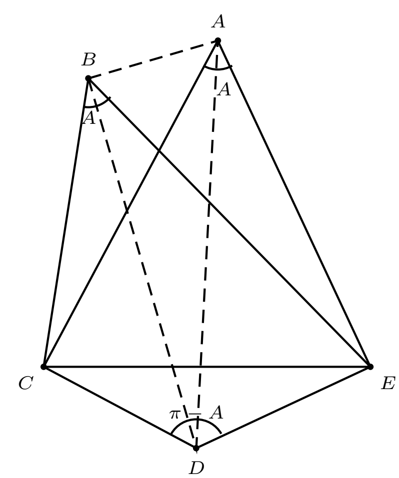

Figure 6

Figure 6 shows one Brahmagupta pentagon. It is easy to see that it must be cyclic. The side AB, the diagonals AD and BD are to be calculated. We apply Ptolemy's theorem successively to $ABCE$, $ACDE$ and $BCDE$. This yields

$$AB = \frac{2023}{17}, \quad AD = \frac{7215}{17}, \quad BD = \frac{6820}{17}.$$

The figure needs 17 times enlargement. The area $ABCDE$ must be integral because it is the disjoint sum of the Brahmagupta quadrilateral $ABCE$ and the Heron triangle ACD.

3.3. *Brahmagupta hexagons.* To construct a Brahmagupta hexagon it is now easy to see that we need at most four Heron triangles taken in any combination from the families (1) and (2). We use the four triangles T_2, T_3, T_5, T_8 to illustrate the hexagon in Figure 7. We leave the calculations to the reader.

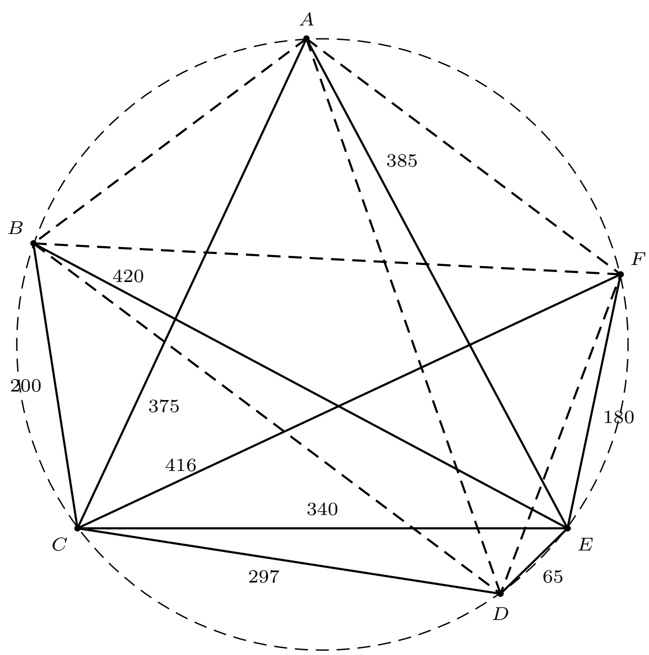

Figure 7

4. Conclusion

In principle the problem of determining Brahmagupta n-gons, $n > 3$, has been solved because all Heron triangle families have been determined by (3) and (4) (in fact by (3) alone). In general to construct a Brahmagupta n-gon, at most $n-2$ Heron triangles taken in any combination from (3) and (4) are needed. They can be adjoined as described in this paper. We pose the following counting problem to the reader.

Given $n-2$ Heron triangles, (i) all from a single family, or (ii) m from one Heron family and the remaining $n-m-2$ from the supplementary angle family, how many Brahmagupta n-gons can be constructed?

It is now natural to conjecture that Heron triangles chosen from appropriate families adjoin to give Heron n-gons. To support this conjecture we give two Heron quadrilaterals generated in this way.

Example 4. From the $\cos\theta = \frac{3}{5}$ family, $7(5,5,6)$ and $6(4,13,15)$ adjoined with $(35,53,24)$ and $6(7,15,20)$ from the supplementary family (with $\cos\theta = -\frac{3}{5}$) to give $ABCD$ with

$$AB = 35,\ BC = 53,\ CD = 78,\ AD = 120,\ AC = 66,\ BD = 125,$$

and area 3300. See Figure 8A.

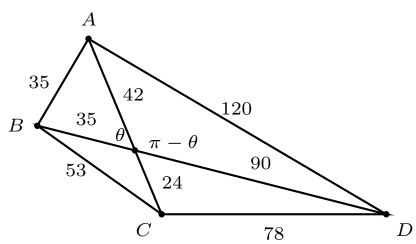

Figure 8A

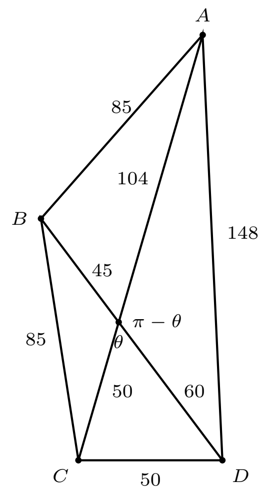

Figure 8B

Example 5. From the same families, the Heron triangles $10(5,5,6)$, $(85,45,104)$ with $5(17,9,10)$ and $4(37,15,26)$ to give a Heron quadrilateral $ABCD$ with

$$AB = 85,\ BC = 85,\ CD = 50,\ AD = 148,\ AC = 154,\ BD = 105,$$

and area 6468. See Figure 8B.

Now, the haunting question is: Which appropriate two members of the θ family adjoin with two appropriate members of the $\pi - \theta$ family to generate Heron quadrilaterals?

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