Forum Geometricorum
Volume 5 (2005) 127–132.

FORUM GEOM
ISSN 1534-1178

Another Proof of van Lamoen's Theorem and Its Converse

Nguyen Minh Ha

Abstract. We give a proof of Floor van Lamoen's theorem and its converse on the circumcenters of the cevasix configuration of a triangle using the notion of directed angle of two lines.

1. Introduction

Let P be a point in the plane of triangle ABC with traces A', B', C' on the sidelines BC, CA, AB respectively. We assume that P does not lie on any of the sidelines. According to Clark Kimberling [1], triangles PCB', $PC'B$, PAC', $PA'C$, PBA', $PB'A$ form the *cevasix configuration* of P. Several years ago, Floor van Lamoen discovered that when P is the centroid of triangle ABC, the six circumcenters of the cevasix configuration are concylic. This was posed as a problem in the *American Mathematical Monthly* and was solved in [2, 3]. In 2003, Alexei Myakishev and Peter Y. Woo [4] gave a proof for the converse, that is, if the six circumcenters of the cevasix configuration are concylic, then P is either the centroid or the orthocenter of the triangle.

In this note we give a new proof, which is quite different from those in [2, 3], of Floor van Lamoen's theorem and its converse, using the directed angle of two lines. Remarkably, both necessity part and sufficiency part in our proof are basically the same. The main results of van Lamoen, Myakishev and Woo are summarized in the following theorem.

Theorem. *Given a triangle ABC and a point P, the six circumcenters of the cevasix configuration of P are concyclic if and only if P is the centroid or the orthocenter of ABC.*

We shall assume the given triangle non-equilateral, and omit the easy case when ABC is equilateral. For convenience, we adopt the following notations used in [4].

Triangle	PCB'	$PC'B$	PAC'	$PA'C$	PBA'	$PB'A$
Notation	$\triangle(A_+)$	$\triangle(A_-)$	$\triangle(B_+)$	$\triangle(B_-)$	$\triangle(C_+)$	$\triangle(C_-)$
Circumcenter	A_+	A_-	B_+	B_-	C_+	C_-

It is easy to see that two of these triangles may possibly share a common circumcenter only when they share a common vertex of triangle ABC.

Publication Date: August 24, 2005. Communicating Editor: Floor van Lamoen.
The author thansk Le Chi Quang of Hanoi, Vietnam for his help in translation and preparation of the article.

2. Preliminary Results

Lemma 1. *Let P be a point not on the sidelines of triangle ABC, with traces B', C' on AC, AB respectively. The circumcenters of triangles APB' and APC' coincide if and only if P lies on the reflection of the circumcircle ABC in the line BC.*

The Proof of Lemma 1 is simple and can be found in [4]. We also omit the proof of the following easy lemma.

Lemma 2. *Given a triangle ABC and M, N on the line BC, we have*

$$\frac{\overline{BC}}{\overline{MN}} = \frac{S[ABC]}{S[AMN]},$$

where $\overline{BC}$ and $\overline{MN}$ denote the signed lengths of the line segments BC and MN, and $S[ABC]$, $S[AMN]$ the signed areas of triangle ABC, and AMN respectively.

Lemma 3. *Let P be a point not on the sidelines of triangle ABC, with traces A', B', C' on BC, AC, AB respectively, and K the second intersection of the circumcircles of triangles PCB' and $PC'B$. The line PK is a symmedian of triangle PBC if and only if A' is the midpoint of BC.*

Proof. Triangles $KB'B$ and KCC' are directly similar (see Figure 1). Therefore,

$$\frac{S[KB'B]}{S[KCC']} = \left(\frac{\overline{B'B}}{\overline{CC'}}\right)^2.$$

On the other hand, by Lemma 2 we have

$$\frac{S[KPB]}{S[KPC]} = \frac{\frac{\overline{PB}}{\overline{B'B}} \cdot S[KB'B]}{\frac{\overline{PC}}{\overline{CC'}} \cdot S[KCC']}.$$

Thus,

$$\frac{S[KPB]}{S[KPC]} = \frac{\overline{PB}}{\overline{PC}}.\frac{\overline{B'B}}{\overline{CC'}}.$$

It follows that PK is a symmedian line of triangle PBC, which is equivalent to the following

$$\frac{S[KPB]}{S[KPC]} = -\left(\frac{\overline{PB}}{\overline{PC}}\right)^2, \quad \frac{\overline{PB}.\overline{B'B}}{\overline{PC}.\overline{CC'}} = -\left(\frac{\overline{PB}}{\overline{PC}}\right)^2, \quad \frac{\overline{B'B}}{\overline{C'C}} = \frac{\overline{PB}}{\overline{PC}}.$$

The last equality is equivalent to $BC \parallel B'C'$, by Thales' theorem, or A' is the midpoint of BC, by Ceva's theorem. □

Remark. Since the lines BC' and CB' intersect at A, the circumcircles of triangles PCB' and $PC'B$ must intersect at two distinct points. This remark confirms the existence of the point K in Lemma 3.

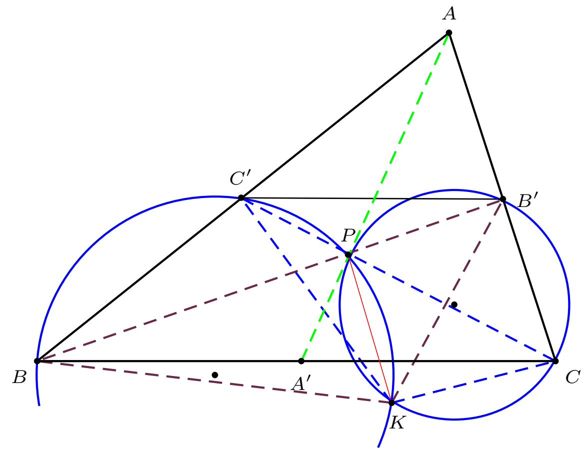

Figure 1

Lemma 4. *Given a triangle XYZ and pairs of points M, N on YZ, P, Q on ZX, and R, S on XY respectively. If the points in each of the quadruples P, Q, R, S; R, S, M, N; M, N, P, Q are concyclic, then all six points M, N, P, Q, R, S are concyclic.*

Proof. Suppose that (O_1), (O_2), (O_3) are the circles passing through the quadruples (P, Q, R, S), (R, S, M, N), and (M, N, P, Q) respectively. If O_1, O_2, O_3 are disctinct points, then YZ, ZX, XY are respectively the radical axis of pairs of circles $(O_2), (O_3)$; $(O_3), (O_1)$; $(O_1), (O_2)$. Hence, YZ, ZX, XY are concurrent, or parallel, or coincident, which is a contradiction. Therefore, two of the three points O_1, O_2, O_3 coincide. It follows that six points M, N, P, Q, R, S are concyclic. □

Remark. In Lemma 4, if $M = N$ and the circumcircles of triangles RSM, MPQ touch YZ at M, then the five points M, P, Q, R, S lie on the same circle that touches YZ at the same point M.

3. Proof of the main theorem

Suppose that perpendicular bisectors of AP, BP, CP bound a triangle XYZ. Evidently, the following pairs of points B_+, C_-; C_+, A_-; A_+, B_- lie on the lines YZ, ZX, XY respectively. Let H and K respectively be the feet of the perpendiculars from P on A_-A_+, B_-B_+ (see Figure 2).

Sufficiency part. If P is the orthocenter of triangle ABC, then $B_+ = C_-$; $C_+ = A_-$; $A_+ = B_-$. Obviously, the six points B_+, C_-, C_+, A_-, A_+, B_- lie on the same circle. If P is the centroid of triangle ABC, then no more than one of the three following possibilities happen: $B_+ = C_-$; $C_+ = A_-$; $A_+ = B_-$, by Lemma 1. Hence, we need to consider two cases.

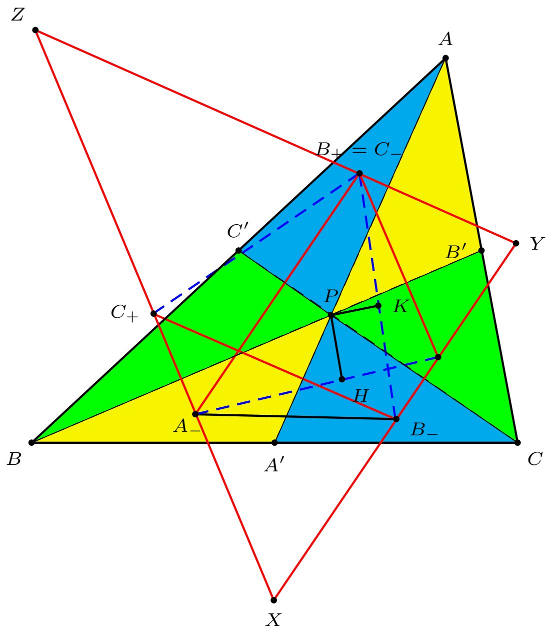

Figure 2

Case 1. Only one of three following possibilities occurs: $B_+ = C_-$, $C_+ = A_-$, $A_+ = B_-$.

Without loss of generality, we may assume that $B_+ = C_-$, $C_+ \neq A_-$ and $A_+ \neq B_-$ (see Figure 2). Since P is the centroid of triangle ABC, A' is the midpoint of the segment BC. By Lemma 3, we have

$$(PH, PB) = (PC, PA') \pmod{\pi}.$$

In addition, since A_-A_+, A_-C_+, B_-A_+, B_-C_+ are respectively perpendicular to PH, PB, PC, PA', we have

$$(A_-A_+, A_-C_+) \equiv (PH, PB) \pmod{\pi}.$$
$$(B_-A_+, B_-C_+) \equiv (PC, PA') \pmod{\pi}.$$

Thus, $(A_-A_+, A_-C_+) \equiv (B_-A_+, B_-C_+) \pmod{\pi}$, which implies that four points C_+, A_-, A_+, B_- are concyclic.

Similarly, we have

$$(PK, PC) = (PA, PB') \pmod{\pi}.$$

Moreover, since B_-B_+, B_-A_+, YZ, B_+A_+ are respectively perpendicular to PK, PC, PA, PB', we have

$$(B_-B_+, B_-A_+) \equiv (PK, PC) \pmod{\pi}.$$
$$(YZ, B_+A_+) \equiv (PA, PB') \pmod{\pi}.$$

Thus, $(B_-B_+, B_-A_+) \equiv (YZ, B_+A_+) \pmod{\pi}$, which implies that the circumcircle of triangle $B_+B_-A_+$ touches YZ at B_+.

The same reasoning also shows that the circumcircle of triangle $B_+C_+A_-$ touches YZ at B_+.

Therefore, the six points B_+, C_-, C_+, A_-, A_+, B_- lie on the same circle and this circle touches YZ at $B_+ = C_-$ by the remark following Lemma 4.

Case 2. None of the three following possibilities occurs: $B_+ = C_-; C_+ = A_-; A_+ = B_-$.

Similarly to case 1, each quadruple of points (C_+, A_-, A_+, B_-), (A_+, B_-, B_+, C_-), (B_+, C_-, C_+, A_-) are concyclic. Hence, by Lemma 4, the six points B_+, C_-, C_+, A_-, A_+, B_- are concyclic.

Necessity part. There are three cases.

Case 1. No less than two of the following possibilities occur: $B_+ = C_-$, $C_+ = A_-$, $A_+ = B_-$.

By Lemma 1, P is the orthocenter of triangle ABC.

Case 2. Only one of the following possibilites occurs: $B_+ = C_-$, $C_+ = A_-$, $A_+ = B_-$. We assume without loss of generality that $B_+ = C_-$, $C_+ \neq A_-$, $A_+ \neq B_-$.

Since the six points B_+, C_-, C_+, A_-, A_+, B_- are on the same circle, so are the four points C_+, A_-, A_+, B_-. It follows that

$$(A_-A_+, A_-C_+) \equiv (B_-A_+, B_-C_+) \pmod{\pi}.$$

Note that lines PH, PB, PC, PA' are respectively perpendicular to A_-A_+, A_-C_+, B_-A_+, B_-C_+. It follows that

$$(PH, PB) \equiv (A_-A_+, A_-C_+) \pmod{\pi}.$$
$$(PC, PA') \equiv (B_-A_+, B_-C_+) \pmod{\pi}.$$

Therefore, $(PH, PB) \equiv (PC, PA') \pmod{\pi}$. Consequently, A' is the midpoint of BC by Lemma 3.

On the other hand, it is evident that $B_+A_- \parallel B_-A_+$; $B_+A_+ \parallel C_+A_-$, and we note that each quadruple of points (B_+, A_-, B_-, A_+), (B_+, A_+, C_+, A_-) are concyclic. Therefore, we have $B_+B_- = A_+A_- = B_+C_+$. It follows that triangle $B_+B_-C_+$ is isosceles with $C_+B_+ = B_+B_-$. Note that YZ passes B_+ and is parallel to C_+B_-, so that we have YZ touches the circle passing six points $B_+ = C_-$, C_+, A_-, A_+, B_- at $B_+ = C_-$. It follows that

$$(B_-B_+, B_-A_+) \equiv (YZ, B_+A_+) \pmod{\pi}.$$

In addition, since PK, PC, PA, PB' are respectively perpendicular to B_-B_+, B_-A_+, YZ, B_+A_+, we have

$$(PK, PC) \equiv (B_-B_+, B_-A_+) \pmod{\pi}.$$
$$(PA, PB') \equiv (YZ, B_+A_+) \pmod{\pi}.$$

Thus, $(PK, PC) \equiv (PA, PB') \pmod{\pi}$. By Lemma 3, B' is the midpoint of CA. We conclude that P is the centroid of triangle ABC.

Case 3. None of the three following possibilities occur: $B_+ = C_-$, $C_+ = A_-$, $A_+ = B_-$.

Similarly to case 2, we can conclude that A', B' are respectively the midpoints of BC, CA. Thus, P is the centroid of triangle ABC.

This completes the proof of the main theorem.

References

[1] C. Kimberling Triangle centers and central triangles, *Congressus Numeratium*, 129 (1998), 1–285

[2] F. M. van Lamoen, Problem 10830, *Amer. Math. Monthly*, 2000 (107) 863; solution by the Monthly editors, 2002 (109) 396–397.

[3] K. Y. Li, Concyclic problems, *Mathematical Excalibur*, 6 (2001) Number 1, 1–2; available at http://www.math.ust.hk/excalibur.

[4] A. Myakishev and Peter Y. Woo, On the Circumcenters of Cevasix Configurations, *Forum Geom.*, 3 (2003) 57–63.

Nguyen Minh Ha: Faculty of Mathematics, Hanoi University of Education, Xuan Thuy, Hanoi, Vietnam

E-mail address: minhha27255@yahoo.com