Forum Geometricorum
Volume 5 (2005) 137–141.

FORUM GEOM
ISSN 1534-1178

On an Erdős Inscribed Triangle Inequality

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Abstract. A comparison between the area of a triangle and that of an inscribed triangle is investigated. The result obtained extend a result of Aassila giving insight into an inequality of P. Erdős.

1. Introduction

Consider a triangle ABC divided into four smaller non-degenerate triangles, a central one $C_1A_1B_1$ inscribed in ABC and three others on the sides of this central triangle, as depicted in

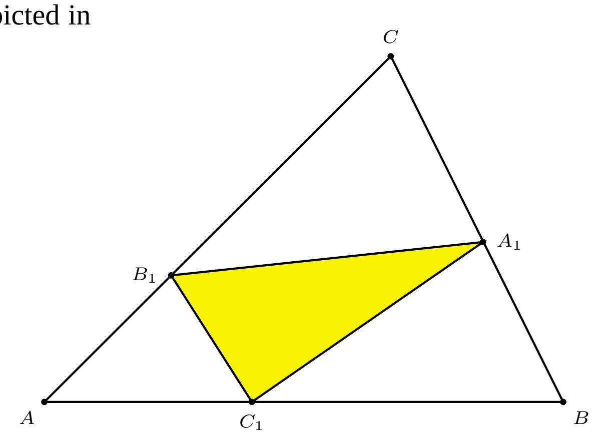

Figure 1

A question with a long history is that of comparing the area of ABC to that of the inscribed triangle $C_1A_1B_1$. In 1956, H. Debrunnner [5] proposed the inequality

$$\mathbf{area}\,(C_1A_1B_1) \geq \min\{\mathbf{area}\,(AC_1B_1),\ \mathbf{area}\,(C_1BA_1),\ \mathbf{area}\,(B_1A_1C)\}\,; \tag{1}$$

according to John Rainwater [7], this inequality originated with P. Erdős and was communicated by N. D. Kazarinoff and J. R. Isbell. However, Rainwater was more precise in stating that $C_1A_1B_1$ cannot have the smallest area of the four unless all four are equal with A_1, B_1, and C_1 the midpoints of the sides BC, CA, and AB.

A proof of (1) first appeared in A. Bager [2] and later in A. Bager [3] and P. H. Diananda [6]. Diananda's proof is particularly noteworthy; in addition to proving Erdős' inequality, it also shows that the stronger form of (1) holds

$$\mathbf{area}\,(C_1A_1B_1) \geq \sqrt{\mathbf{area}\,(AC_1B_1)\cdot\mathbf{area}\,(C_1BA_1)} \tag{2}$$

where, without loss of generality, it is assumed that

$$0 < \mathbf{area}\,(AC_1B_1) \leq \mathbf{area}\,(C_1BA_1) \leq \mathbf{area}\,(B_1A_1C)\,.$$

Publication Date: September 28, 2005. Communicating Editor: Paul Yiu.

The purpose of this paper is to show that a sharper inequality is possible when more care is placed in choosing the points $A_1,\ B_1$ and C_1. In so doing we extend Aassila's inequality [1]:

$$4\cdot\mathbf{area}\,(A_1B_1C_1)\le\mathbf{area}\,(ABC),$$

which is valid when these points are chosen so as to partition the perimeter of ABC into equal length segments. Our main result is

Theorem 1. *Let ABC be a triangle, and let $A_1,\ B_1,\ C_1$ be on $BC,\ CA,\ AB$, respectively, with none of $A_1,\ B_1,\ C_1$ coinciding with a vertex of ABC. If*

$$\frac{AB+BA_1}{AC+CA_1}=\frac{BC+CB_1}{AB+AB_1}=\frac{AC+AC_1}{BC+BC_1}=\alpha,$$

then

$$4\cdot\mathbf{area}\,(A_1B_1C_1)\le\mathbf{area}\,(ABC)+s^4\left(\frac{\alpha-1}{\alpha+1}\right)^2\cdot\mathbf{area}\,(ABC)^{-1}$$

where s is the semi-perimeter of ABC.

When $\alpha=1$ we obtain Aassila's result.

Corollary 2 (Aassila [1])**.** *Let ABC be a triangle, and let $A_1,\ B_1,\ C_1$ be on $BC,\ CA,\ AB$, respectively, with none of $A_1,\ B_1,\ C_1$ coinciding with a vertex of ABC. If*

$$\begin{aligned}AB+BA_1&=AC+CA_1,\\BC+CB_1&=AB+AB_1,\\AC+AC_1&=BC+BC_1,\end{aligned}$$

then

$$4\cdot\mathbf{area}\,(A_1B_1C_1)\le\mathbf{area}\,(ABC)\,.$$

2. Proof of Theorem 1

We shall make use of the following two lemmas.

Lemma 3 (Curry [4])**.** *For any triangle ABC, and standard notation,*

$$4\sqrt{3}\cdot\mathbf{area}\,(ABC)\le\frac{9abc}{a+b+c}.\tag{3}$$

Equality holds if and only if $a=b=c$.

Lemma 4. *For any triangle ABC, and standard notation,*

$$\min\{a^2+b^2+c^2,\ ab+bc+ca\}\ge4\sqrt{3}\cdot\mathbf{area}\,(ABC)\,.\tag{4}$$

To prove Theorem 1, we begin by computing the area of the corner triangle AC_1B_1:

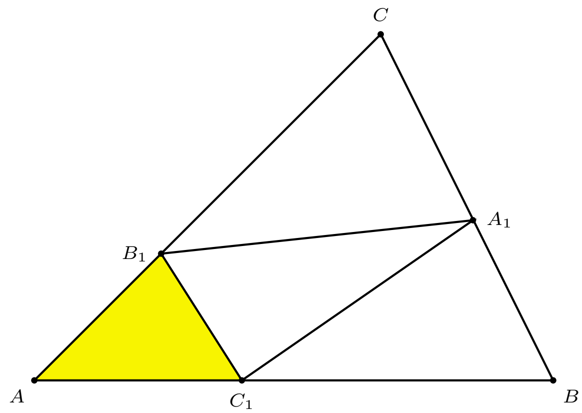

Figure 2

then

$$\begin{aligned}\mathbf{area}\,(AC_1B_1) &= \frac{1}{2}AC_1\cdot AB_1\cdot\sin A\\ &= \frac{1}{2}AC_1\cdot AB_1\cdot\frac{2\cdot\mathbf{area}\,(ABC)}{AB\cdot AC}\\ &= \frac{AC_1}{AB}\cdot\frac{AB_1}{AC}\cdot\mathbf{area}\,(ABC)\,.\end{aligned}$$

For the semi-perimeter s of ABC we have

$$\begin{aligned}2s &= AB+BC+AC\\ &= (AB+AB_1)+(BC+CB_1)\\ &= (\alpha+1)(c+AB_1),\end{aligned}$$

and

$$AB_1=\frac{2}{\alpha+1}s-c$$

where $c=AB$. Also,

$$\begin{aligned}2s &= AB+BC+AC\\ &= (AC+AC_1)+(BC+BC_1)\\ &= \left(1+\frac{1}{\alpha}\right)(AC+AC_1)\\ &= \frac{\alpha+1}{\alpha}(b+AC_1),\end{aligned}$$

and

$$AC_1=\frac{2\alpha}{\alpha+1}s-b$$

with $b = AC$. Hence

$$\mathbf{area}\,(AC_1B_1) = \frac{1}{bc}\left(\frac{2\alpha}{\alpha+1}s - b\right)\left(\frac{2}{\alpha+1}s - c\right)\cdot\mathbf{area}\,(ABC)\,. \tag{5}$$

Similar computations yield

$$\mathbf{area}\,(C_1BA_1) = \frac{1}{ca}\left(\frac{2\alpha}{\alpha+1}s - c\right)\left(\frac{2}{\alpha+1}s - a\right)\cdot\mathbf{area}\,(ABC), \tag{6}$$

and

$$\mathbf{area}\,(B_1A_1C) = \frac{1}{ab}\left(\frac{2\alpha}{\alpha+1}s - a\right)\left(\frac{2}{\alpha+1}s - b\right)\cdot\mathbf{area}\,(ABC)\,. \tag{7}$$

From these formulae,

$$\begin{aligned}
&\mathbf{area}\,(A_1B_1C_1)\\
=&\,\mathbf{area}\,(ABC) - \mathbf{area}\,(AC_1B_1) - \mathbf{area}\,(C_1BA_1) - \mathbf{area}\,(B_1A_1C)\\
=&\left[1 - \frac{1}{bc}\left(\frac{2\alpha}{\alpha+1}s - b\right)\left(\frac{2}{\alpha+1}s - c\right) - \frac{1}{ca}\left(\frac{2\alpha}{\alpha+1}s - c\right)\left(\frac{2}{\alpha+1}s - a\right)\right.\\
&\quad\left. - \frac{1}{ab}\left(\frac{2\alpha}{\alpha+1}s - a\right)\left(\frac{2}{\alpha+1}s - b\right)\right]\cdot\mathbf{area}\,(ABC)\\
=&\frac{1}{abc}\left[\left(\frac{2}{\alpha+1}s - a\right)\left(\frac{2}{\alpha+1}s - b\right)\left(\frac{2}{\alpha+1}s - c\right)\right.\\
&\quad\left. + \left(\frac{2\alpha}{\alpha+1}s - a\right)\left(\frac{2\alpha}{\alpha+1}s - b\right)\left(\frac{2\alpha}{\alpha+1}s - c\right)\right]\cdot\mathbf{area}\,(ABC)\,.
\end{aligned}$$

But

$$\begin{aligned}
&\left(\frac{2}{\alpha+1}s - a\right)\left(\frac{2}{\alpha+1}s - b\right)\left(\frac{2}{\alpha+1}s - c\right)\\
+&\left(\frac{2\alpha}{\alpha+1}s - a\right)\left(\frac{2\alpha}{\alpha+1}s - b\right)\left(\frac{2\alpha}{\alpha+1}s - c\right)\\
=&2(s-a)(s-b)(s-c) + 2\left(\frac{\alpha-1}{\alpha+1}\right)^2 s^3\\
=&\frac{2}{s}\left[\mathbf{area}\,(ABC)\right]^2 + 2\left(\frac{\alpha-1}{\alpha+1}\right)^2 s^3.
\end{aligned}$$

Hence

$$\frac{abc\cdot s}{2}\cdot\mathbf{area}\,(A_1B_1C_1) = \left[\mathbf{area}\,(ABC)\right]^3 + s^4\cdot\left(\frac{\alpha-1}{\alpha+1}\right)^2\cdot\mathbf{area}\,(ABC)\,. \tag{8}$$

From (3) and (4)

$$\begin{aligned}\frac{abc\cdot s}{2} &\geq \frac{\sqrt{3}}{9}\cdot(a+b+c)^2\cdot\mathbf{area}\,(ABC)\\ &\geq \frac{\sqrt{3}}{9}[a^2+b^2+c^2+2(ab+bc+ca)]\cdot\mathbf{area}\,(ABC)\\ &\geq \frac{\sqrt{3}}{9}\cdot 12\sqrt{3}\cdot\mathbf{area}\,(ABC)^2\\ &\geq 4\cdot\mathbf{area}\,(ABC)^2.\end{aligned}$$

Finally, from (8)

$$\begin{aligned}&4\cdot\mathbf{area}\,(ABC)^2\cdot\mathbf{area}\,(A_1B_1C_1)\\ \leq\ &\frac{abc\cdot s}{2}\cdot\mathbf{area}\,(A_1B_1C_1)\\ \leq\ &[\mathbf{area}\,(ABC)]^3+s^4\cdot\left(\frac{\alpha-1}{\alpha+1}\right)^2\cdot\mathbf{area}\,(ABC)\end{aligned}$$

and a division by $\mathbf{area}\,(ABC)^2$ produces

$$4\cdot\mathbf{area}\,(A_1B_1C_1)\leq\mathbf{area}\,(ABC)+s^4\cdot\left(\frac{\alpha-1}{\alpha+1}\right)^2\cdot[\mathbf{area}\,(ABC)]^{-1}$$

completing the proof of the theorem.

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