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The Eppstein Centers and the Kenmotu Points

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Abstract. The Kenmotu points of a triangle are triangle centers associated with squares each with a pair of opposite vertices on two sides of a triangle. Given a triangle ABC, we prove that the Kenmotu points of the intouch triangle are the same as the Eppstein centers associated with the Soddy circles of ABC.

1. Introduction

D. Eppstein [1] has discovered two interesting triangle centers associated with the Soddy circles of a triangle. Given a triangle ABC, construct three circles with centers at A, B, C, mutually tangent to each other externally at T_a, T_b, T_c respectively. These are indeed the points of tangency of the incircle of triangle ABC, and triangle $T_aT_bT_c$ is the intouch triangle of ABC. The inner (respectively outer) Soddy circle is the circle (S) (respectively (S')) tangent to each of these circles externally at S_a, S_b, S_c (respectively internally at S'_a, S'_b, S'_c).

Theorem 1 (Eppstein [1])**.** (1) *The lines T_aS_a, T_bS_b, and T_cS_c are concurrent at a point M.*
(2) *The lines $T_aS'_a$, $T_bS'_b$, and $T_cS'_c$ are concurrent at a point M'.*

See Figures 1 and 2. In [2], M and M' are the Eppstein centers X_{481} and X_{482}. Eppstein showed that these points are on the line joining the incenter I to the Gergonne point $G_{\rm e}$. See Figure 1.

The Kenmotu points of a triangle, on the other hand, are associated with triads of congruent squares. Given a triangle ABC, the Kenmotu point $K_{\rm e}$ is the unique point such that there are congruent squares $K_{\rm e}B_cA_aC_b$, $K_{\rm e}C_aB_bA_c$, and $K_{\rm e}A_bC_cB_a$ with the same orientation as triangle ABC, and with A_b, A_c on BC, B_c, B_a on CA, and C_a, C_b on AB respectively. We call $K_{\rm e}$ the positive Kenmotu point. There is another triad of congruent squares with the opposite orientation as ABC, sharing a common vertex at the negative Kenmuto point $K'_{\rm e}$. See Figure 3. These Kenmotu points lie on the Brocard axis of triangle ABC, which contains the circumcenter O and the symmedian point K.

The intouch triangle $T_aT_bT_c$ has circumcenter I and symmedian point $G_{\rm e}$. It is immediately clear that the Kenmotu points of the intouch triangle lie on the same

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line as do the Soddy and Eppstein centers of triangle ABC. The main result of this note is the following theorem.

Theorem 2. *The positive and negative Kenmotu points of the intouch triangle $T_aT_bT_c$ coincide with the Eppstein centers M and M'.*

We shall give two proofs of this theorem.

2. The Eppstein centers

According to [2], the coordinates of the Eppstein centers were determined by E. Brisse. [1] We shall work with homogeneous barycentric coordinates and make use of standard notations in triangle geometry. In particular, r_a, r_b, r_c denote the radii of the respective excircles, and S stands for twice the area of the triangle.

Theorem 3. *The homogeneous barycentric coordinates of the Eppstein centers are*
(1) $M = (a + 2r_a : b + 2r_b : c + 2r_c)$, *and*
(2) $M' = (a - 2r_a : b - 2r_b : c - 2r_c)$.

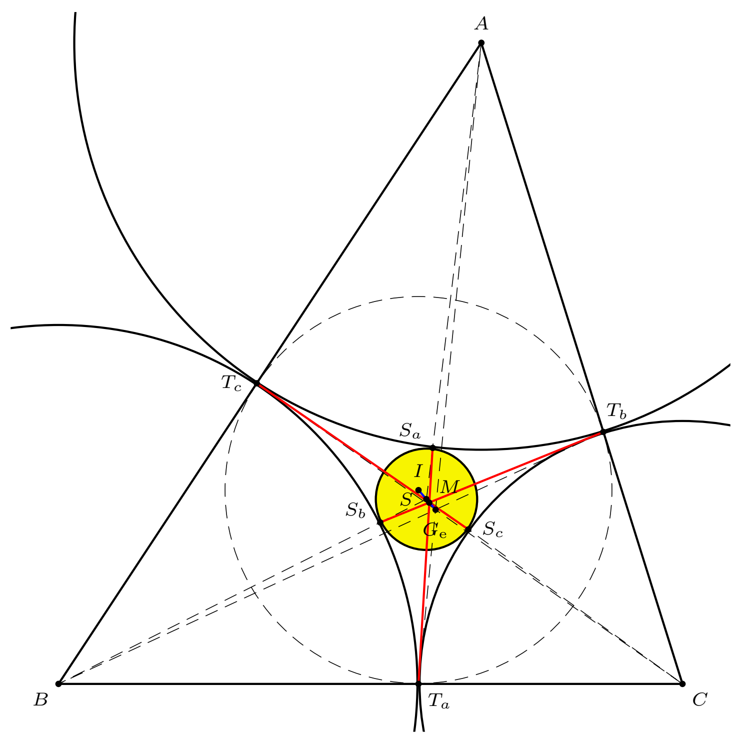

Figure 1. The Soddy center S and the Eppstein center M

[1]The coordinates of X_{481} and X_{482} in [2] (September 2005 edition) should be interchanged.

Remark. In [2], the Soddy centers appear as $X_{175} = S'$ and $X_{176} = S$. In homogeneous barycentric coordinates

$$S =(a + r_a : b + r_b : c + r_c),$$
$$S' =(a - r_a : b - r_b : c - r_c).$$

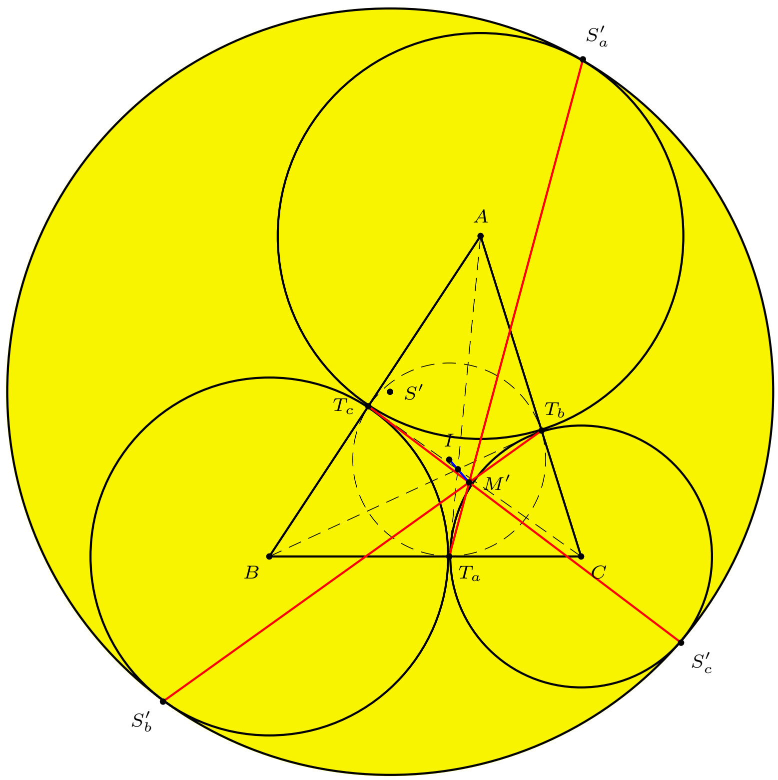

Figure 2. The Soddy center S' and the Eppstein center M'

3. The Kenmotu points

The Kenmotu points $K_{\rm e}$ and $K'_{\rm e}$ have homogeneous barycentric coordinates $(a^2(S_A \pm S) : b^2(S_B \pm S) : c^2(S_C \pm S))$. They are therefore points on the Brocard axis OK. See Figure 3.

Proposition 4. *The Kenmotu points* $K_{\rm e}$ *and* $K'_{\rm e}$ *divide the segment* OK *in the ratio*

$$OK_{\rm e} : K_{\rm e}K =a^2 + b^2 + c^2 : 2S,$$
$$OK'_{\rm e} : K'_{\rm e}K =a^2 + b^2 + c^2 : -2S.$$

Proof. A typical point on the Brocard axis has coordinates

$$K^*(\theta) = (a^2(S_A + S_\theta) : b^2(S_B + S_\theta) : c^2(S_C + S_\theta)).$$

It divides the segment OK in the ratio

$$OK^*(\theta) : K^*(\theta)K = (a^2 + b^2 + c^2)\sin\theta : 2S \cdot \cos\theta.$$

The Kenmotu points are the points $K_{\rm e}$ and $K'_{\rm e}$ are the points $K^*(\theta)$ for $\theta = \frac{\pi}{4}$ and $-\frac{\pi}{4}$ respectively. □

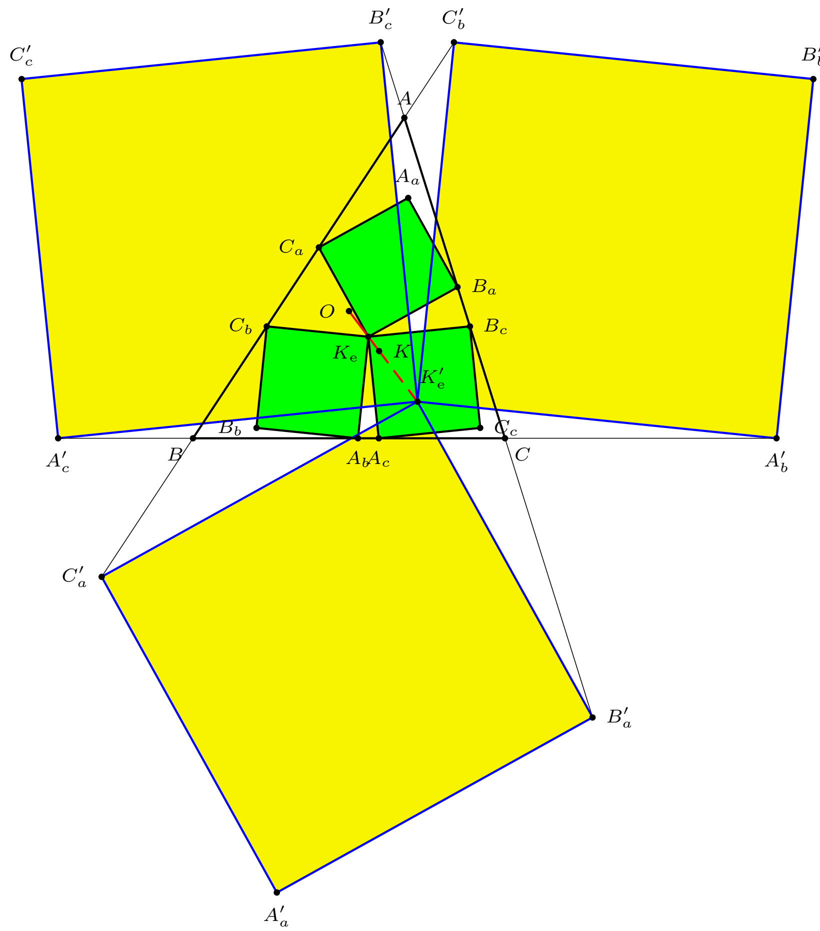

Figure 3. The Kenmotu points $K_{\rm e}$ and $K'_{\rm e}$

4. First proof of Theorem 2

We shall make use of the following results.

Lemma 5. (1) $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{s}{4R}$.
(2) $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = \frac{4R+r}{2R}$.
(3) $r_a + r_b + r_c = 4R + r$.

The intouch triangle $T_aT_bT_c$ has sidelengths

$$T_bT_c = 2r\cos\frac{A}{2}, \qquad T_cT_a = 2r\cos\frac{B}{2}, \qquad T_aT_b = 2r\cos\frac{C}{2}.$$

The area of the intouch triangle is

$$\frac{1}{2}\overline{S} = \frac{1}{2}T_cT_a \cdot T_aT_b \cdot \sin T_a = 2r^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 2r^2 \cdot \frac{s}{4R}.$$

On the other hand,

$$T_bT_c^2 + T_cT_a^2 + T_aT_b^2 = 4r^2\left(\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2}\right) = \frac{2r^2(4R+r)}{R}.$$

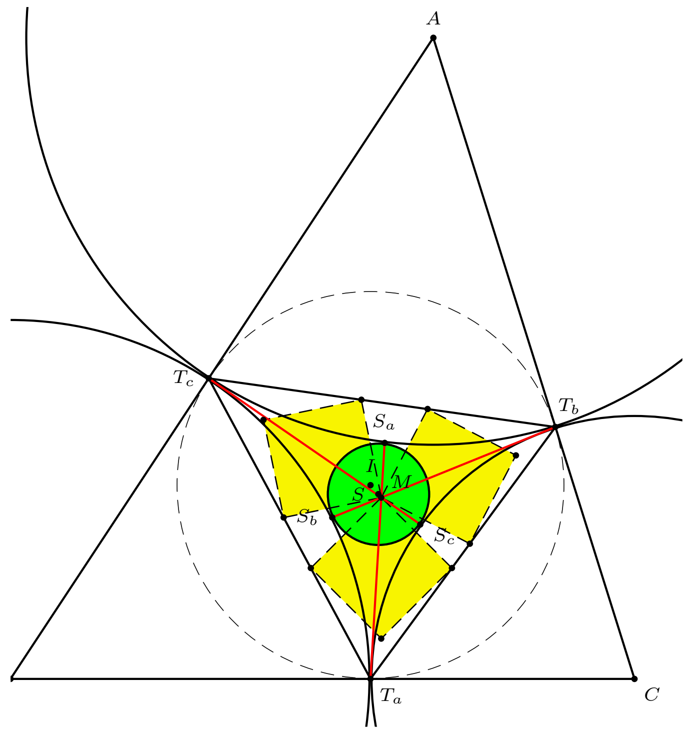

Figure 4. The positive Kenmotu point of the intouch triangle

By Proposition 4, the positive Kenmotu point $\overline{K}_{\rm e}$ of the intouch triangle divides the segment $IG_{\rm e}$ in the ratio

$$\begin{aligned} I\overline{K}_{\rm e} : \overline{K}_{\rm e}G_{\rm e} =& T_bT_c^2 + T_cT_a^2 + T_aT_b^2 : 2\overline{S} \\ =& 4R + r : s \\ =& r_a + r_b + r_c : s. \end{aligned}$$

It has absolute barycentric coordinates

$$\begin{aligned}\overline{K}_{\rm e} =&\frac{1}{s+r_a+r_b+r_c}\left(s\cdot I+(r_a+r_b+r_c)\cdot G_{\rm e}\right)\\ =&\frac{1}{s+r_a+r_b+r_c}\left(\frac{1}{2}(a,b,c)+(r_a,r_b,r_c)\right)\\ =&\frac{1}{2(s+r_a+r_b+r_c)}\cdot(a+2r_a,b+2r_b,c+2r_c).\end{aligned}$$

Therefore, $\overline{K}_{\rm e}$ has homogeneous barycentric coordinates $(a+2r_a : b+2r_b : +2r_c)$. By Theorem 3, it coincides with the Eppstein center M. See Figure 4.

Similar calculations show that the Eppstein center M' coincides with the negative Kenmotu point $\overline{K}'_{\rm e}$ of the intouch triangle. See Figure 5. The proof of Theorem 2 is now complete.

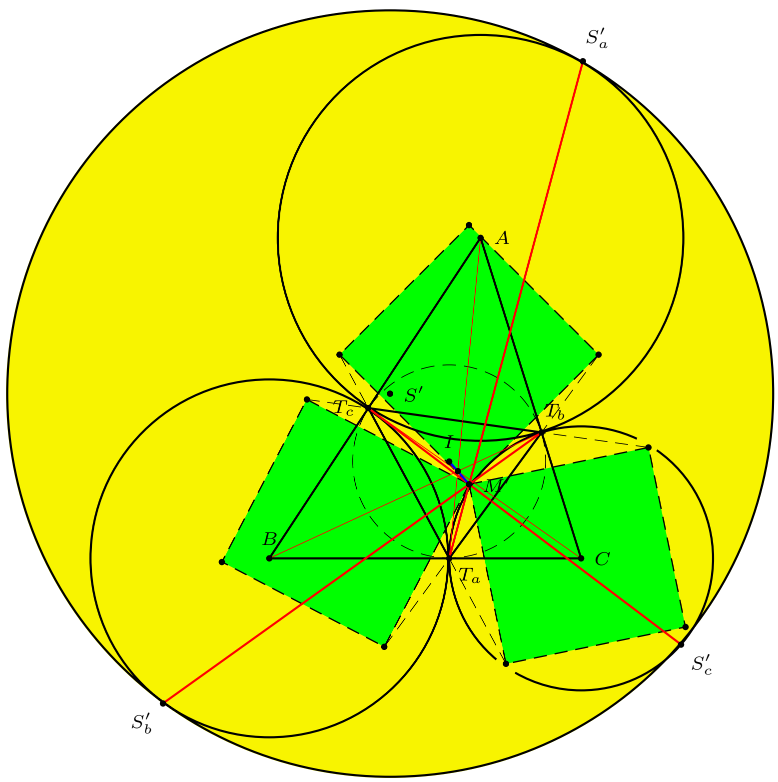

Figure 5. The negative Kenmotu point of the intouch triangle

5. Second proof of Theorem 2

Consider a point P with homogeneous barycentric coordinates $(u' : v' : w')$ with respect to the intouch triangle $T_aT_bT_c$. We determine its coordinates with respect to the triangle ABC. By the definition of barycentric coordinates, a system of three masses u', v' and w' at the points T_a, T_b and T_c will balance at P. The mass u' at T_a can be replaced by a mass $\frac{s-c}{a} \cdot u'$ at B and a mass $\frac{s-b}{a} \cdot u'$ at C. Similarly, the mass v' at T_b can be replaced by a mass $\frac{s-a}{b} \cdot v'$ at C and a mass $\frac{s-c}{b} \cdot v'$ at A, and the mass w' at T_c by a mass $\frac{s-b}{c} \cdot w'$ at A and a mass $\frac{s-a}{c} \cdot w'$ at B. The resulting mass at A is therefore

$$\frac{s-c}{b} \cdot v' + \frac{s-b}{c} \cdot w' = \frac{a(c(s-c)v' + b(s-b)w')}{abc}.$$

From similar expressions for the masses at B and C, we obtain

$$(a(c(s-c)v'+b(s-b)w') : b(a(s-a)w'+c(s-c)u') : c(b(s-b)u'+a(s-a)v'))$$

for the barycentric coordinates of P with respect to ABC.

The Kenmotu point K_{e} appears the triangle center X_{371} in [2]. For the Kenmotu point of the intouch triangle, we may take

$$\begin{aligned} u' =& T_bT_c(\cos T_a + \sin T_a) \\ =& 2(s-a)\sin\frac{A}{2}\left(\sin\frac{A}{2} + \cos\frac{A}{2}\right), \\ v' =& 2(s-b)\sin\frac{B}{2}\left(\sin\frac{B}{2} + \cos\frac{B}{2}\right), \\ w' =& 2(s-c)\sin\frac{C}{2}\left(\sin\frac{C}{2} + \cos\frac{C}{2}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} u =& a(c(s-c)v' + b(s-b)w') \\ =& 2a(s-b)(s-c)\left(c\cdot\sin\frac{B}{2}\left(\sin\frac{B}{2}+\cos\frac{B}{2}\right) + b\cdot\sin\frac{C}{2}\left(\sin\frac{C}{2}+\cos\frac{C}{2}\right)\right) \\ =& 2a(s-b)(s-c)\left(c\sin^2\frac{B}{2} + b\sin^2\frac{C}{2} + c\cdot\frac{\sin B}{2} + b\cdot\frac{\sin C}{2}\right) \\ =& 2a(s-b)(s-c)\left(c\cdot\frac{(s-c)(s-a)}{ca} + b\cdot\frac{(s-a)(s-b)}{ab} + \frac{bc}{2R}\right) \\ =& 2(s-a)(s-b)(s-c)\left(a + \frac{abc}{2R(s-a)}\right) \\ =& 2(s-a)(s-b)(s-c)\left(a + \frac{S}{s-a}\right) \\ =& 2(s-a)(s-b)(s-c)(a+2r_a). \end{aligned}$$

Similar expressions for v and w give

$$u : v : w = a + 2r_a : b + 2r_b : c + r_c,$$

which are the coordinates of the Eppstein center M.

References

[1] D. Eppstein, Tangent spheres and triangle centers, *Amer. Math. Monthly*, 108 (2001) 63–66.
[2] C. Kimberling, *Encyclopedia of Triangle Centers*, available at
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