Forum Geometricorum
Volume 6 (2006) 25–28.

FORUM GEOM
ISSN 1534-1178

A Note on the Droz-Farny Theorem

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Abstract. We give a simple characterization of the Droz-Farny pairs of lines through a point of the plane.

In [3] J-P. Erhmann and F. van Lamoen prove a projective generalization of the Droz-Farny line theorem. They say that a pair of lines (l, l') is a *pair of DF-lines through a point P with respect to a given triangle ABC* if they intercept the line BC in the points X and X', CA in Y and Y', and AB in Z and Z' in such a way that the midpoints of the segments XX', YY', and ZZ' are collinear. They then prove that (l, l') is a pair of DF-lines if and only if l and l' are tangent lines of a parabola inscribed in ABC (see also [5]). Thus, the DF-lines through P are the pairs of conjugate lines in the involution $\mathcal{I}$ determined by the lines through P that are tangent to the parabolas of the pencil of parabolas inscribed in ABC. Through a general point P, there passes just one orthogonal pair of DF-lines with respect to ABC; call this pair the ODF-lines through P with respect to ABC.

Considering the tangent lines through P at the three degenerate inscribed parabolas of ABC, it also follows that (PA, line through P parallel with BC), (PB, line parallel through P with CA), and (PC, line parallel through P with AB), are three conjugate pairs of lines of the involution $\mathcal{I}$.

Recall that the medial triangle $A'B'C'$ of ABC is the triangle whose vertices are the midpoints of $BC, CA,$ and AB, and that the anticomplementary triangle $A''B''C''$ of ABC is the triangle whose medial triangle is ABC ([4]).

Theorem. *A pair (l, l') of lines is a pair of DF-lines through P with respect to ABC, if and only if (l, l') is a pair of conjugate diameters of the conic $\mathcal{C}_P$ with center P, circumscribed at the anticomplementary triangle $A''B''C''$ of ABC. In particular, the ODF-lines through P are the axes of this conic.*

Proof. Since A is the midpoint of $B''C''$, and $B''C''$ is parallel with BC, it follows immediately that PA, and the line through P, parallel with BC, are conjugate diameters of the conic $\mathcal{C}_P$. In the same way, PB and the line through P parallel with CA (and PC and the line trough P parallel with AB) are also conjugate

Publication Date: January 30, 2006. Communicating Editor: J. Chris Fisher.

diameters of $\mathcal{C}_P$. Since two pairs of corresponding lines determine an involution, this completes the proof. $\square$

Remark that the orthocenter H of ABC is the center of the circumcircle of the anticomplementary triangle $A''B''C''$. Since any two orthogonal diameters of a circle are conjugate, we find by this special case the classical Droz-Farny theorem: Perpendicular lines through H are DF-pairs with respect to triangle ABC.

As a corollary of this theorem, we can characterize the axes of any circumscribed ellipse or hyperbola of ABC as the ODF-lines through its center with regard to the medial triangle $A'B'C'$ of ABC. And in the same way we can construct the axes of any circumscribed ellipse or hyperbola of any triangle, associated with ABC.

Examples

1. The Jerabek hyperbola of ABC (the isogonal conjugate of the Euler line of ABC) is the rectangular hyperbola through A, B, C, the orthocenter H, the circumcenter O and the Lemoine (or symmedian) point K of ABC, and its center is Kimberling center X_{125} with trilinear coordinates $(bc(b^2+c^2-a^2)(b^2-c^2)^2, .., ..)$, which is a point of the nine-point circle of ABC (the center of any circumscribed rectangular hyperbola is on the nine-point circle). The axes of this hyperbola are the ODF-lines through X_{125}, with respect to the medial triangle $A'B'C'$ of ABC.

2. The Kiepert hyperbola of ABC is the rectangular hyperbola through A, B, C, H, the centroid G of ABC, and through the Spieker center (the incenter of the medial triangle of ABC). It has center X_{115} with trilinear coordinates $(bc(b^2-c^2)^2, .., ..)$ on the nine-point circle. Its axes are the ODF-lines through X_{115} with respect to the medial triangle $A'B'C'$.

3. The Steiner ellipse of ABC is the circumscribed ellipse with center the centroid G of ABC. It is homothetic to (and has the same axes of) the Steiner ellipses of the medial triangle $A'B'C'$ and of the anticomplementary triangle $A''B''C''$ of ABC. These axes are the ODF-lines through G with respect to ABC (and to $A'B'C'$, and to $A''B''C''$).

4. The Feuerbach hyperbola is the rectangular hyperbola through A, B, C, H, the incenter I of ABC, the Mittenpunkt (the symmedian point of the excentral triangle $I_AI_BI_C$, where I_A, I_B, I_C are the excenters of ABC), with center the Feuerbach point F (at which the incircle and the nine-point circle are tangent; trilinear coordinates $(bc(b-c)^2(b+c-a), .., ..))$. Its axes are the ODF-lines through F, with respect to the medial triangle $A'B'C'$ of ABC.

5. The Stammler hyperbola of ABC has trilinear equation

$$(b^2-c^2)x_1^2 + (c^2-a^2)x_2^2 + (a^2-b^2)x_3^2 = 0.$$

It is the rectangular hyperbola through the incenter I, the excenters I_A, I_B, I_C, the circumcenter O, and the symmedian point K. It is also the Feuerbach hyperbola of the tangential triangle of ABC, and its center is the focus of the Kiepert parabola (inscribed parabola with directrix the Euler line of ABC), which is Kimberling center X_{110} with trilinear coordinates $(\frac{a}{b^2-c^2}, .., ..)$, on the circumcircle of ABC, which is the nine-point circle of the excentral triangle $I_AI_BI_C$. The axes of this Stammler hyperbola are the ODF-lines through X_{110}, with regard to the medial triangle of $I_AI_BI_C$.

Remark that center X_{110} is the fourth common point (apart from A, B, and C) of the conic through A, B, C, and with center the symmedian point K of ABC, which has trilinear equation

$$a(-a^2+b^2+c^2)x_2x_3 + b(a^2-b^2+c^2)x_3x_1 + c(a^2+b^2-c^2)x_1x_2 = 0,$$

and the circumcircle of ABC.

6. The conic with trilinear equation

$$a^2(b^2-c^2)x_1^2 + b^2(c^2-a^2)x_2^2 + c^2(a^2-b^2)x_3^2 = 0$$

is the rectangular hyperbola through the incenter I, through the excenters I_A, I_B, I_C, and through the centroid G of ABC. It is also circumscribed to the anticomplementary triangle $A''B''C''$ (recall that the trilinear coordinates of A'', B'', and C'' are $(-bc, ac, ab), (bc, -ac, ab)$, and $(bc, ac, -ab)$, respectively). Its center is the Steiner point X_{99} with trilinear coordinates $(\frac{bc}{b^2-c^2}, \frac{ca}{c^2-a^2}, \frac{ab}{a^2-b^2})$, a point of intersection of the Steiner ellipse and the circumcircle of ABC.

Remark that the circumcircle of ABC is the nine-point circle of $A''B''C''$ and also of $I_AI_BI_C$. It follows that the axes of this hyperbola, which is often called the Wallace or the Steiner hyperbola, are the ODF-lines through the Steiner point X_{99} with regard to ABC, and also with regard to the medial triangle of the excentral triangle $I_AI_BI_C$.

Remarks. (1) A biographical note on Arnold Droz-Farny can be found in [1].

(2) A generalization of the Droz-Farny theorem in the three-dimensional Euclidean space was given in an article by J. Bilo [2].

(3) Finally, we give a construction for the ODF-lines through a point P with respect to the triangle ABC, *i.e.*, for the orthogonal conjugate pair of lines through P of the involution $\mathcal{I}$ in the pencil of lines through P, determined by the conjugate pairs (PA, line l_a through P parallel to BC) and (PB, line l_b through P parallel to CA): intersect a circle $\mathcal{C}$ through P with these conjugate pairs:

$$\begin{aligned} \mathcal{C} \cap PA &= Q, & \qquad \mathcal{C} \cap l_a &= Q', \\ \mathcal{C} \cap PB &= R, & \qquad \mathcal{C} \cap l_b &= R', \end{aligned}$$

then (Q, Q') and (R, R') determine an involution $\mathcal{I}'$ on $\mathcal{C}$, with center $QQ' \cap RR' = T$. Each line through T intersect the circle $\mathcal{C}$ in two conjugate points of $\mathcal{I}'$. In particular, the line through T and through the center of $\mathcal{C}$ intersects $\mathcal{C}$ in two points S and S', such that PS and PS' are the orthogonal conjugate pair of lines of the involution $\mathcal{I}$.

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