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The Locations of Triangle Centers

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Abstract. The orthocentroidal circle of a non-equilateral triangle has diameter GH where G is the centroid and H is the orthocenter. We show that the Fermat, Gergonne and symmedian points are confined to, and range freely over the interior disk punctured at its center. The Mittenpunkt is also confined to and ranges freely over another punctured disk, and the second Fermat point is confined to and ranges freely over the exterior of the orthocentroidal circle. We also show that the circumcenter, centroid and symmedian point determine the sides of the reference triangle ABC.

1. Introduction

All results concern non-equilateral non-degenerate triangles. The orthocentroidal circle $\mathcal{S}_{GH}$ has diameter GH, where G is the centroid and H is the orthocenter of triangle ABC. Euler showed [3] that O, G and I determine the sides a, b and c of triangle ABC. Here O denotes the circumcenter and I the incenter. Later Guinand [4] showed that I ranges freely over the open disk $\mathcal{D}_{GH}$ (the interior of $\mathcal{S}_{GH}$) punctured at the nine-point center N. This work involved showing that certain cubic equations have real roots. Recently Smith [9] showed that both results can be achieved in a straightforward way; that I can be anywhere in the punctured disk follows from Poncelet's porism, and a formula for IG^2 means that the position of I in $\mathcal{D}_{GH}$ enables one to write down a cubic polynomial which has the side lengths a, b and c as roots. As the triangle ABC varies, the Euler line may rotate and the distance GH may change. In order to say that I ranges freely over all points of this punctured open disk, it is helpful to rescale by insisting that the distance GH is constant; this can be readily achieved by dividing by the distance GH or OG as convenient. It is also helpful to imagine that the Euler line is fixed.

In this paper we are able to prove similar results for the symmedian (K), Fermat (F) and Gergonne (G_{e}) points, using the same disk $\mathcal{D}_{GH}$ but punctured at its midpoint J rather than at the nine-point center N. We show that, O, G and K determine a, b and c. The Morleys [8] showed that O, G and the first Fermat point F determine the reference triangle by using complex numbers. We are not able to show that O, G and G_{e} determine a, b and c, but we conjecture that they do.

Since I, G, S_{p} and N_{a} are collinear and spaced in the ratio $2 : 1 : 3$ it follows from Guinand's theorem [4] that the Spieker center and Nagel point are confined

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to, and range freely over, certain punctured open disks, and each in conjunction with O and G determines the triangle's sides. Since G_e, G and M are collinear and spaced in the ratio $2:1$ it follows that M ranges freely over the open disk on diameter OG with its midpoint deleted. Thus we now know how each of the first ten of Kimberling's triangle centers [6] can vary with respect to the scaled Euler line.

Additionally we observe that the orthocentroidal circle forms part of a coaxal system of circles including the circumcircle, the nine-point circle and the polar circle of the triangle. We give an areal descriptions of the orthocentroidal circle. We show that the Feuerbach point must lie outside the circle $\mathcal{S}_{GH}$, a result foreshadowed by a recent internet announcement. This result, together with assertions that the symmedian and Gergonne points (and others) must lie in or outside the orthocentroidal disk were made in what amount to research announcements on the Yahoo message board Hyacinthos [5] on 27th and 29th November 2004 by M. R. Stevanovic, though his results do not yet seem to be in published form. Our results were found in March 2005 though we were unaware of Stevanovic's announcement at the time.

The two Brocard points enjoy the *Brocard exclusion principle*. If triangle ABC is not isosceles, exactly one of the Brocard points is in $\mathcal{D}_{GH}$. If it is isosceles, then both Brocard points lie on the circle $\mathcal{S}_{GH}$. This last result was also announced by Stevanovic.

The fact that the (first) Fermat point must lie in the punctured disk $\mathcal{D}_{GH}$ was established by Várilly [10] who wrote ... *this suggests that the neighborhood of the Euler line may harbor more secrets than was previously known*. We offer this article as a verification of this remark.

We realize that some of the formulas in the subsequent analysis are a little daunting, and we have had recourse to the use of the computer algebra system DERIVE from time to time. We have also empirically verified our geometric formulas by testing them with the CABRI geometry package; when algebraic formulas and geometric reality co-incide to 9 decimal places it gives confidence that the formulas are correct. We recommend this technique to anyone with reason to doubt the algebra.

We suggest [1], [2] and [7] for general geometric background.

2. The orthocentroidal disk

This is the interior of the circle on diameter GH and a point X lies in the disk if and only if $\angle GXH > \frac{\pi}{2}$. It will lie on the boundary if and only if $\angle GXH = \frac{\pi}{2}$. These conditions may be combined to give

$$\overline{XG} \cdot \overline{XH} \leq 0, \tag{1}$$

with equality if and only if X is on the boundary.

In what follows we initially use Cartesian vectors with origin at the circumcenter O, with $\overline{OA} = \mathbf{x}, \overline{OB} = \mathbf{y}, \overline{OC} = \mathbf{z}$ and, taking the circumcircle to have radius 1, we have

$$|\mathbf{x}| = |\mathbf{y}| = |\mathbf{z}| = 1 \tag{2}$$

and

$$\mathbf{y}\cdot\mathbf{z} = \cos 2A = \frac{a^4+b^4+c^4-2a^2(b^2+c^2)}{2b^2c^2} \tag{3}$$

with similar expressions for $\mathbf{z}\cdot\mathbf{x}$ and $\mathbf{x}\cdot\mathbf{y}$ by cyclic change of a, b and c. This follows from $\cos 2A = 2\cos^2 A - 1$ and the cosine rule.

We take X to have position vector

$$\frac{u\mathbf{x}+v\mathbf{y}+w\mathbf{z}}{u+v+w},$$

so that the unnormalised areal co-ordinates of X are simply (u, v, w). Now

$$3\overline{XG} = \frac{((v+w-2u),(w+u-2v),(u+v-2w))}{u+v+w},$$

not as areals, but as components in the $\mathbf{x},\mathbf{y},\mathbf{z}$ frame and

$$\overline{XH} = \frac{(v+w, w+u, u+v)}{u+v+w}.$$

Multiplying by $(u+v+w)^2$ we find that condition (1) becomes

$$\sum_{\text{cyclic}} \{(v+w-2u)(v+w)\}$$
$$+\sum_{\text{cyclic}} [(w+u-2v)(u+v)+(w+u)(u+v-2w)]\mathbf{y}\cdot\mathbf{z} \le 0,$$

where we have used (2). The sum is taken over cyclic changes. Next, simplifying and using (3), we obtain

$$\sum_{\text{cyclic}} 2(u^2+v^2+w^2-vw-wu-uv)(a^2b^2c^2)$$
$$+\sum_{\text{cyclic}} (u^2-v^2-w^2+vw)[a^2(a^4+b^4+c^4)-2a^4(b^2+c^2)].$$

Dividing by $(a+b+c)(b+c-a)(c+a-b)(a+b-c)$ the condition that $X(u, v, w)$ lies in the disk $\mathcal{D}_{GH}$ is

$$(b^2+c^2-a^2)u^2+(c^2+a^2-b^2)v^2+(a^2+b^2-c^2)w^2$$
$$-a^2vw-b^2wu-c^2uv < 0 \tag{4}$$

and the equation of the circular boundary is

$$S_{GH} \equiv (b^2+c^2-a^2)x^2+(c^2+a^2-b^2)y^2+(a^2+b^2-c^2)z^2$$
$$-a^2yz-b^2zx-c^2xy = 0. \tag{5}$$

The polar circle has equation

$$S_P \equiv (b^2+c^2-a^2)x^2+(c^2+a^2-b^2)y^2+(a^2+b^2-c^2)z^2 = 0.$$

The circumcircle has equation

$$S_C \equiv a^2yz+b^2zx+c^2xy = 0.$$

The nine-point circle has equation

$$S_N \equiv (b^2+c^2-a^2)x^2+(c^2+a^2-b^2)y^2+(a^2+b^2-c^2)z^2 \\ -2a^2yz-2b^2zx-2c^2xy=0.$$

Evidently $S_{GH}-S_C=S_P$ and $S_N+2S_C=S_P$. We have established the following result.

Theorem 1. *The orthocentroidal circle forms part of a coaxal system of circles including the circumcircle, the nine-point circle and the polar circle of the triangle.*

It is possible to prove the next result by calculating that $JK<OG$ directly (recall that J is the midpoint of GH), but it is easier to use the equation of the orthocentroidal circle.

Theorem 2. *The symmedian point lies in the disc $\mathcal{D}_{GH}$.*

Proof. Substituting $u=a^2, v=b^2, w=c^2$ in the left hand side of equation (4) we get $a^4b^2+b^4c^2+c^4a^2+b^4a^2+c^4b^2+a^4c^2-3a^2b^2c^2-a^6-b^6-c^6$ and this quantity is negative for all real a,b,c except $a=b=c$. This follows from the well known inequality for non-negative l,m and n that

$$l^3+m^3+n^3+3lmn \geq \sum_{sym} l^2m$$

with equality if and only if $l=m=n$. □

We offer a second proof. The line AK with areal equation $c^2y=b^2z$ meets the circumcircle of ABC at D with co-ordinates $(-a^2,2b^2,2c^2)$, with similar expressions for points E and F by cyclic change. The reflection D' of D in BC has co-ordinates $(a^2,b^2+c^2-a^2,b^2+c^2-a^2)$ with similar expressions for E' and F'. It is easy to verify that these points lie on the orthocentroidal disk by substituting in (5) (the circle through D',E' and F' is the Hagge circle of K).

Let $\mathbf{d}',\mathbf{e}'$ and $\mathbf{f}'$ denote the vector positions D',E' and F' respectively. It is clear that

$$\mathbf{s}=(2b^2+2c^2-a^2)\mathbf{d}'+(2c^2+2a^2-b^2)\mathbf{e}'+(2a^2+2b^2-c^2)\mathbf{f}'$$

but $2b^2+2c^2-a^2=b^2+c^2+2bc\cos A\geq(b-c)^2>0$ and similar results by cyclic change. Hence relative to triangle $D'E'F'$ all three areal co-ordinates of K are positive so K is in the interior of triangle $D'E'F'$ and hence inside its circumcircle. We are done.

The incenter lies in $\mathcal{D}_{GH}$. Since $IGN_{\rm a}$ are collinear and $IG:GN_{\rm a}=1:2$ it follows that Nagel's point is outside the disk. However, it is instructive to verify these facts by substituting relevant areal co-ordinates into equation (5), and we invite the interested reader to do so.

Theorem 3. *One Brocard point lies in $\mathcal{D}_{GH}$ and the other lies outside $\mathcal{S}_{GH}$, or they both lie simultaneously on $\mathcal{S}_{GH}$* (which happens if and only if the reference triangle is isosceles).

Proof. Let $f(u,v,w)$ denote the left hand side of equation (4). One Brocard point has unnormalised areal co-ordinates

$$(u,v,w) = (a^2b^2, b^2c^2, c^2a^2)$$

and the other has unnormalised areal co-ordinates

$$(p,q,r) = (a^2c^2, b^2a^2, c^2b^2),$$

but they have the same denominator when normalised. It follows that $f(u,v,w)$ and $f(p,q,r)$ are proportional to the powers of the Brocard points with respect to $\mathcal{S}_{GH}$ with the same constant of proportionality. If the sum of these powers is zero we shall have established the result. This is precisely what happens when the calculation is made. □

The fact that the Fermat point lies in the orthocentroidal disk was established recently [10] by Várilly.

Theorem 4. *Gergonne's point lies in the orthocentroidal disk* $\mathcal{D}_{GH}$.

Proof. Put $u = (c+a-b)(a+b-c)$, $v = (a+b-c)(b+c-a)$, $w = (b+c-a)(c+a-b)$ and the left hand side of (5) becomes

$$-18a^2b^2c^2 + \sum_{\text{cyclic}} \left(-a^5(b+c) + 4a^4(b^2-bc+b^2) - 6b^3c^3 + 5a^3(b^2c+bc^2)^2\right)$$

which we want to show is negative. This is not immediately recognisable as a known inequality, but performing the usual trick of putting $a = m+n$, $b = n+l$, $c = l+m$ where $l,m,n > 0$ we get the required inequality (after division by 8) to be

$$2(m^3n^3 + n^3l^3 + l^3m^3) > lmn \left(\sum_{sym} m^2n\right)$$

where the final sum is over all possible permutations and l,m,n not all equal. Now $l^3(m^3+n^3) > l^3(m^2n+mn^2)$ and adding two similar inequalities we are done. Equality holds if and only if $a=b=c$, which is excluded. □

3. The determination of the triangle sides.

3.1. *The symmedian point.* We will find a cubic polynomial which has roots a^2, b^2, c^2 given the positions of O, G and K.

The idea is to express the formulas for OK^2, GK^2 and JK^2 in terms of $u = a^2+b^2+c^2$, $v^2 = a^2b^2+b^2c^2+c^2a^2$ and $w^3 = a^2b^2c^2$.

We first note some equations which are the result of routine calculations.

$$16[ABC]^2 = (a+b+c)(b+c-a)(c+a-b)(a+b-c)$$
$$= \sum_{\text{cyclic}} (2a^2b^2 - a^4) = 4v^2 - u^2.$$

It is well known that the circumradius R satisfies the equation $R = \frac{abc}{4[ABC]}$ so

$$R^2 = \frac{a^2b^2c^2}{16[ABC]^2} = \frac{w^3}{(4v^2 - u^2)}.$$

$$OG^2 = \frac{1}{9a^2b^2c^2}\left[\left(\sum_{\text{cyclic}} a^6\right) + 3a^2b^2c^2 - \left(\sum_{sym} a^4b^2\right)\right] R^2$$

$$= \frac{u^3 + 9w^3 - 4uv^2}{9(4v^2 - u^2)} = \frac{w^3}{(4v^2 - u^2)} - \frac{u}{9} = R^2 - \frac{a^2 + b^2 + c^2}{9}.$$

By areal calculations one may obtain the formulas

$$OK^2 = \frac{4R^2 \sum_{\text{cyclic}} \left(a^4 - a^2b^2\right)}{(a^2 + b^2 + c^2)^2} = \frac{4w^3(u^2 - 3v^2)}{u^2(4v^2 - u^2)},$$

$$GK^2 = \frac{\left(\sum_{\text{cyclic}} 3a^4(b^2 + c^2)\right) - 15a^2b^2c^2 - \left(\sum_{\text{cyclic}} a^6\right)}{(a^2 + b^2 + c^2)^2} = \frac{6uv^2 - u^3 - 27w^3}{9u^2},$$

$$JK^2 = OG^2\left(1 - \frac{48[ABC]^2}{(a^2 + b^2 + c^2)^2}\right) = \frac{4(u^3 + 9w^3 - 4uv^2)(u^2 - 3v^2)}{9u^2(4v^2 - u^2)}.$$

The full details of the last calculation will be given when justifying (14).

Note that

$$\frac{OK^2}{JK^2} = \frac{9w^3}{(u^3 + 9w^3 - 4uv^2)}$$

or

$$\frac{JK^2}{OK^2} = 1 - \frac{u(4v^2 - u^2)}{9w^3}.$$

We simplify expressions by putting $u = p$, $4v^2 - u^2 = q$ and $w^3 = r$. We have

$$OG^2 = \frac{r}{q} - \frac{p}{9}. \tag{6}$$

Now $u^2 - 3v^2 = -\frac{3}{4}(4v^2 - u^2) + \frac{1}{4}u^2 = -\frac{3}{4}q + \frac{1}{4}p^2$ so

$$OK^2 = 4r\frac{(\frac{1}{4}p^2 - \frac{3}{4}q)}{p^2q} = \frac{(p^2 - 3q)r}{p^2q} = \frac{r}{q} - \frac{3r}{p^2} = r\left(\frac{1}{q} - \frac{3}{p^2}\right) \tag{7}$$

Also $6v^2 - u^2 = \frac{3}{2}(4v^2 - u^2) + \frac{1}{2}u^2 = \frac{3q}{2} + \frac{p^2}{2}$

$$GK^2 = \frac{p(3q/2 + p^2/2) - 27r}{9p^2} = \frac{p}{18} + \frac{q}{6p} - \frac{3r}{p^2} \tag{8}$$

$$\frac{OK^2}{JK^2} = 1 - \frac{pq}{9r} \tag{9}$$

We now have four quantities that are homogeneous of degree 1 in a^2, b^2 and c^2. These are $p, q/p, r/q, r/p^2 = x, y, z, s$ respectively, where $xs = r/p = yz$. We have (6) $OG^2 = z - x/9$, (7) $OK^2 = z - 3s$, (8) $GK^2 = x + 6y - 3s$ and (9) $\frac{OK^2}{JK^2} = 1 - x/(9z)$ or $9zOK^2 = (9z - x)JK^2$. Now u, v and w are known unambiguously and hence the equations determine a^2, b^2 and c^2 and therefore a, b and c.

3.2. *The Fermat point.* We assume that O, G and the (first) Fermat point are given. Then F determines and is determined by the second Fermat point F' since they are inverse in $\mathcal{S}_{GH}$. In pages 206-208 [8] the Morleys show that O, G and F determine triangle ABC using complex numbers.

4. Filling the disk

Following [9], we fix R and r, and consider the configuration of Poncelet's porism for triangles. This diagram contains a fixed circumcircle, a fixed incircle, and a variable triangle ABC which has the given circumcircle and incircle. Moving point A towards the original point B by sliding it round the circumcircle takes us continuously through a family of triangles which are pairwise not directly similar (by angle and orientation considerations) until A reaches the original B, when the starting triangle is recovered, save that its vertices have been relabelled. Moving through triangles by sliding A to B in this fashion we call a *Poncelet cycle*.

We will show shortly that for X the Fermat, Gergonne or symmedian point, passage through a Poncelet cycle takes X round a closed path arbitrarily close to the boundary of the orthocentroidal disk scaled to have constant diameter. By choosing the neighbourhood of the boundary sufficiently small, it follows that X has winding number 1 (with suitable orientation) with respect to J as we move through a Poncelet cycle.

We will show that when r approaches $R/2$ (as we approach the equilateral configuration) a Poncelet cycle will keep X arbitrarily close to, but never reaching, J in the scaled orthocentroidal disk. Moving the ratio r/R from close to 0 to close to $1/2$ induces a homotopy between the 'large' and 'small' closed paths. So the small path also has winding number 1 with respect to J. One might think it obvious that every point in the scaled punctured disk must arise as a possible X on a closed path intermediate between a path sufficiently close to the edge and a path sufficiently close to the deletion. There are technical difficulties for those who seek them, since we have not eliminated the possibility of exotic paths. However, a rigorous argument is available via complex analysis. Embed the scaled disk in the complex plane. Let γ be an anticlockwise path (i.e. winding number $+1$) near the boundary and δ be an anticlockwise path (winding number also 1) close to the puncture. Suppose (for contradiction) that the complex number z_0 represents a point between the wide path γ and the tight path δ which is not a possible location for X.

The function defined by $1/(z - z_0)$ is meromorphic in $\mathcal{D}_{GH}$ and is analytic save for a simple pole at z_0. However by our hypothesis we have a homotopy of paths from γ to δ which does not involve z_0 being on an intermediate path. Therefore

$$1 = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0} = \frac{1}{2\pi i}\int_\delta \frac{dz}{z - z_0} = 0.$$

Thus $1 = 0$ and we have the required contradiction.

5. Close to the edge

The areal coordinates of the incenter and the symmedian point of triangle ABC are (a,b,c) and (a^2,b^2,c^2) respectively. We consider the mean square distance of the vertex set to itself, weighted once by (a,b,c) and once by (a^2,b^2,c^2). Note that $a,b,c>0$ so $\sigma_I^2, \sigma_S^2 > 0$. The GPAT [9] asserts that

$$\sigma_I^2 + KI^2 + \sigma_K^2 = 2\frac{abc}{a+b+c}\frac{ab+bc+ca}{a^2+b^2+c^2} \leq 4Rr.$$

It follows that $SI < 2\sqrt{Rr}$.

In what follows we fix R and investigate what we can achieve by choosing r to be sufficiently small.

Since I lies in the critical disk we have $OH > OI$ so

$$OH^2 > OI^2 = R^2 - 2Rr.$$

By choosing $r < R/8$ say, we force $9GJ^2 = OH^2 > 3R^2/4$ so $GJ > R\sqrt{3}/6$. Now we have

$$\frac{KI}{GJ} < \frac{2\sqrt{Rr}}{R\sqrt{3}/6} = 4\sqrt{\frac{3r}{R}}. \tag{10}$$

For any $\varepsilon > 0$, there is $K_1 > 0$ so that if $0 < r < K_1$, then $\frac{KI}{GJ} < \frac{\varepsilon}{2}$. Observe that we are dividing by GJ to scale the orthocentroidal disk so that it has fixed radius.

Recall that a passage round a Poncelet cycle induces a path for I in the scaled critical disk which is a circle of Apollonius with defining points O and N with ratio $IO : IN = 2\sqrt{\frac{R}{R-2r}}$. It is clear from the theory of Apollonius circles that there is $K_2 > 0$ such that if $0 < r < K_2$, then $1 - \frac{IJ}{GJ} < \varepsilon/2$.

Now choosing r such that $0 < r < \min\{R/8, K_1, K_2\}$ we have

$$1 - \frac{IJ}{GJ} < \frac{\varepsilon}{2} \tag{11}$$

and $SI/GJ < \frac{\varepsilon}{2}$ so by the triangle inequality

$$\frac{IJ}{GJ} < \frac{KJ}{GJ} + \frac{\varepsilon}{2}. \tag{12}$$

Adding equations (11) and (12) and rearranging we deduce that

$$1 - \varepsilon < \frac{KJ}{GJ}.$$

This shows that for sufficiently small r, the path of K in the scaled critical disk (as the triangle moves through a Poncelet cycle) will be confined to a region at most ε from the boundary. Moreover, assuming that $\varepsilon < 1$ the winding number of K about J will increase by 1, because that it is what happens to I, and J moves in proximity to I.

A similar result holds for the Gergonne point G_e. This is the intersection of the Cevians joining triangle vertices to the opposite contact point of the incircle. The Gergonne point must therefore be inside the incircle.

As before we consider the case that R is fixed. Now $G_eI < r$. We proceed as in the argument for the symmedian point. We get a new version of equation (10) which is

$$\frac{G_eI}{GJ} < \frac{r}{R\sqrt{3}/6} = \frac{2r\sqrt{3}}{R} \tag{13}$$

For any $\varepsilon > 0$, there is a possibly new $K_1 > 0$ so that if $0 < r < K_1$, then $\frac{G_eI}{GJ} < \frac{\varepsilon}{2}$. The rest of the argument proceeds unchanged.

6. Near the orthocentroidal center

6.1. *The symmedian point.* For the purposes of the following calculation only, we will normalize so that $R = 1$. We have

$$\overline{OK} = \frac{a^2\mathbf{x} + b^2\mathbf{y} + c^2\mathbf{z}}{a^2+b^2+c^2}$$

so

$$\begin{aligned}\overline{KJ} &= \sum_{\text{cyclic}} \left(\frac{2}{3} - \frac{a^2}{a^2+b^2+c^2}\right)\mathbf{x} = \frac{\sum_{\text{cyclic}}(2b^2+2c^2-a^2)\mathbf{x}}{3(a^2+b^2+c^2)} \\ &= l\mathbf{x} + m\mathbf{y} + n\mathbf{z}\end{aligned}$$

where l, m and n can be read off.

We have

$$\begin{aligned}a^2b^2c^2KJ^2 &= a^2b^2c^2\left(l^2+m^2+n^2+\sum_{\text{cyclic}} 2mn\mathbf{y}\cdot\mathbf{z}\right) \\ &= (l^2+m^2+n^2)(a^2b^2c^2) + \sum_{\text{cyclic}} mn(a^2(a^4+b^4+c^4) - 2a^4(b^2+c^2)) \\ &= \frac{4P_{10}}{9(a^2+b^2+c^2)^2}\end{aligned}$$

where

$$P_{10} = \sum_{\text{cyclic}} a^{10} - 2a^8(b^2+c^2) + a^6(b^4+4b^2c^2+c^4) - 3a^4b^4c^2.$$

Now

$$OG^2 = \frac{1}{9a^2b^2c^2}\left[\left(\sum_{\text{cyclic}} a^6\right) + 3a^2b^2c^2 - \left(\sum_{sym} a^4b^2\right)\right]$$

so we define Q_6 by

$$OG^2 = \frac{Q_6}{9a^2b^2c^2}.$$

We have $9a^2b^2c^2OG^2 = Q_6$ and

$$9a^2b^2c^2KJ^2 = \frac{4P_{10}}{(a^2+b^2+c^2)^2}$$

so that

$$\frac{OG^2}{KJ^2} = \frac{Q_6(a^2+b^2+c^2)^2}{4P_{10}}.$$

Now a computer algebra (DERIVE) aided calculation reveals that

$$\frac{Q_6(a^2+b^2+c^2) - 4P_{10}}{3(a+b-c)(b+c-a)(c+a-b)(a+b+c)} = Q_6$$

It follows that

$$\frac{KJ^2}{OG^2} = 1 - \frac{48[ABC]^2}{(a^2+b^2+c^2)^2}. \tag{14}$$

Our convenient simplification that $R = 1$ can now be dropped, since the ratio on the left hand side of (14) is dimensionless. As the triangle approaches the equilateral, KJ/OG approaches 0. Therefore in the orthocentroidal disk scaled to have diameter 1, the symmedian point approaches the center J of the circle.

6.2. *The Gergonne point.* Fix the circumcircle of a variable triangle ABC. We consider that the case that r approaches $R/2$, so the triangle ABC approaches (but does not reach) the equilateral. Drop a perpendicular ID to BC.

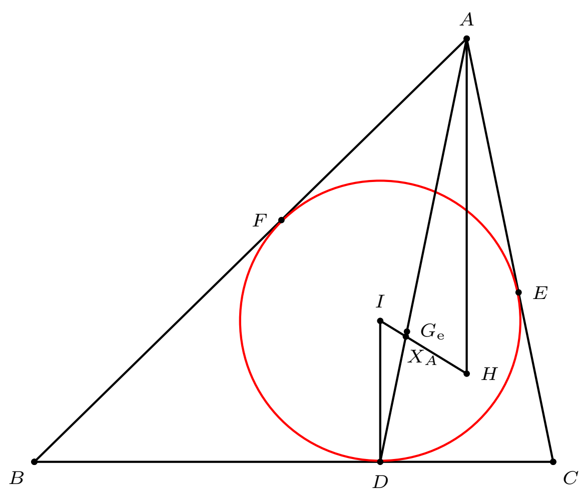

Figure 1

Let AD meet IH at X_A. Triangles IDX_A and HAX_A are similar. When r approaches $R/2$, H approaches O so HA approaches $OA = R$. It follows that $IX_A : X_AH$ approaches $1 : 2$. Similar results hold for corresponding points X_B and X_C.

If we rescale so that points O, G and H are fixed, the points X_A, X_B and X_C all converge to a point X on IH such that $IX : XH = 1 : 2$. Consider the three rays AX_A, BX_B and CX_C which meet at the Gergonne point of the triangle. As ABC approaches the equilateral, these three rays become more and more like the diagonals of a regular hexagon. In particular, if the points X_A, X_B and X_C arise

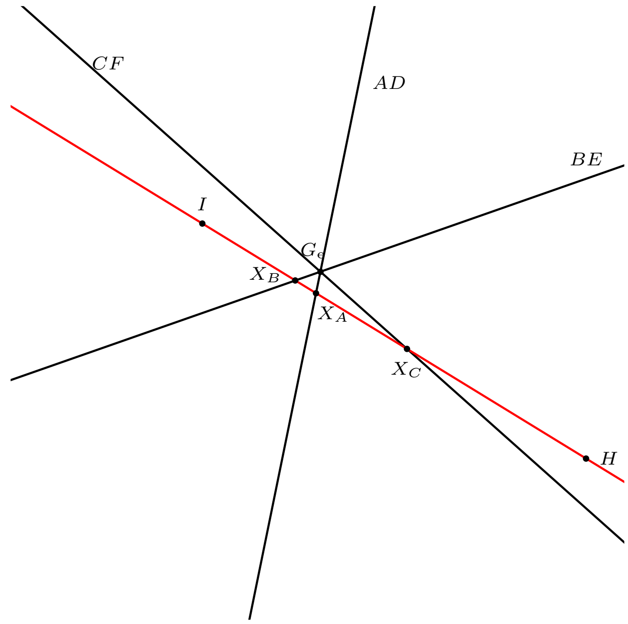

Figure 2

(without loss of generality) in that order on the directed line IH, then $\angle X_A G_e X_C$ is approaching $2\pi/3$, and so we may take this angle to be obtuse. Therefore G_e is inside the circle on diameter $X_A X_C$. Thus in the scaled diagram G_e approaches X, but I approaches N, so G_e approaches the point J which divides NH internally in the ratio $1 : 2$. Thus G_e converges to J, the center of the orthocentroidal circle.

Thus the symmedian and Gergonne points are confined to the orthocentroidal disk, make tight loops around its center J, as well as wide passages arbitrarily near its boundary (as moons of I). Neither of them can be at J for non-equilateral triangles (an easy exercise). By continuity we have proved the following result.

Theorem 5. *Each of the Gergonne and symmedian points are confined to, and range freely over the orthocentroidal disk punctured at its center.*

6.3. *The Fermat point.* An analysis of areal co-ordinates reveals that the Fermat point F lies on the line JK between J and K, and

$$\frac{JF}{FK} = \frac{a^2 + b^2 + c^2}{4\sqrt{3}[ABC]}. \tag{15}$$

From this it follows that as we approach the equilateral limit, F approaches the midpoint of JK. However we have shown that K approaches J in the scaled diagram, so F approaches J.

As r approaches 0 with R fixed, the area $[ABC]$ approaches 0, so in the scaled diagram F can be made arbitrarily close to K (uniformly). It follows that F performs closed paths arbitrarily close to the boundary.

Here is an outline of the areal algebra. The first normalized areal co-ordinate of H is

$$\frac{(c^2+a^2-b^2)(a^2+b^2-c^2)}{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4}$$

and the other co-ordinates are obtained by cyclic changes. We suppress this remark in the rest of our explanation. Since J is the midpoint of GH the first areal co-ordinate of J is

$$\frac{a^4-2b^4-2c^4+b^2c^2+c^2a^2+a^2c^2}{3(a+b+c)(b+c-a)(c+a-b)(a+b-c)}.$$

One can now calculate the areal equation of the line JK as

$$\sum_{\text{cyclic}}(b^2-c^2)(a^2-b^2-bc-c^2)(a^2-b^2+bc-c^2)x=0$$

To calculate the areal co-ordinates of F we first assume that the reference triangle has each angle less that $2\pi/3$. In this case the rays AF, BF and CF meet at equal angles, and one can use trigonometry to obtain a formula for the areal co-ordinates which, when expressed in terms of the reference triangle sides, turns out to be correct for arbitrary triangles. One can either invoke the charlatan's *principle of permanence of algebraic form*, or analyze what happens when a reference angle exceeds $2\pi/3$. In the latter event, the trigonometry involves a sign change dependent on the region in which F lies, but the final formula for the co-ordinates remains unchanged. In such a case, of course, not all of the areal co-ordinates are positive.

The unnormalized first areal co-ordinate of F turns out to be

$$8\sqrt{3}a^2[ABC]+2a^4-4b^4-4c^4+2a^2b^2+2a^2c^2+8b^2c^2.$$

The first areal component K_x of K is

$$\frac{a^2}{a^2+b^2+c^2}$$

and the first component J_x of J is

$$\frac{a^4-2b^4-2c^4+a^2b^2+a^2c^2+4b^2c^2}{48[ABC]^2}.$$

Hence the first co-ordinate of F is proportional to

$$8\sqrt{3}K_x[ABC](a^2+b^2+c^2)+96J_x[ABC]^2.$$

This is linear in J_x and K_x, and the other co-ordinates are obtained by cyclic change. It follows that J, F, K are collinear (as is well known) but also that by the section theorem, JF/FK is given by (15).

The Fermat point cannot be at J in a non-equilateral triangle because the second Fermat point is inverse to the first in the orthocentroidal circle.

We have therefore established the following result.

Theorem 6. *Fermat's point is confined to, and ranges freely over the orthocentroidal disk punctured at its center and the second Fermat point ranges freely over the region external to* $\mathcal{S}_{GH}$.

7. The Feuerbach point

Let F_{e} denote the Feuerbach point.

Theorem 7. *The point F_{e} is always outside the orthocentroidal circle.*

Proof. Let J be the center of the orthocentroidal circle, and N be its nine-point center. In [9] it was established that

$$IJ^2 = OG^2 - \frac{2r}{3}(R - 2r).$$

We have $IN = R/2 - r$, $IF_{\mathrm{e}} = r$ and $JN = OG/2$. We may apply Stewart's theorem to triangle $JF_{\mathrm{e}}N$ with Cevian JI to obtain

$$JF_{\mathrm{e}}^2 = OG^2\left(\frac{2R - r}{2R - 4r}\right) - \frac{rR}{6}. \tag{16}$$

This leaves the issue in doubt so we press on.

$$JF_{\mathrm{e}}^2 = OG^2 + OG^2\left(\frac{3r}{2R - 4r}\right) - \frac{rR}{6}.$$

Now I must lie in the orthocentroidal disk so $IO/3 < OG$ and therefore

$$JF_{\mathrm{e}}^2 > OG^2 + \frac{3rR(R - 2r)}{18(R - 2r)} - \frac{rR}{6} = OG^2.$$

□

Corollary 8. *The positions of I and F_{e} reveal that the interior of the incircle intersects both $\mathcal{D}_{GH}$ and the region external to $\mathcal{S}_{GH}$ non-trivially.*

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