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The Locations of the Brocard Points

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Abstract. We fix and scale to constant size the orthocentroidal disk of a variable non-equilateral triangle ABC. We show that the set of points of the plane which can be either type of Brocard point consists of the interior of the orthocentroidal disk. We give the locus of points which can arise as a Brocard point of specified type, and describe this region and its boundary in polar terms. We show that ABC is determined by the locations of the circumcenter, the centroid and the Brocard points. In some circumstances the location of one Brocard point will suffice.

1. Introduction

For geometric background we refer the reader to [1], [3] and [4]. In [2] and [5] we demonstrated that scaling the orthocentroidal circle (on diameter GH) to have fixed diameter and studying where other major triangle centers can lie relative to this circle is a fruitful exercise. We now address the Brocard points. We consider non-equilateral triangles ABC. We will have occasion to use polar co-ordinates with origin the circumcenter O. We use the Euler line as the reference ray, with OG of length 1. We will describe points, curves and regions by means of polar co-ordinates (r, θ). To fix ideas, the equation of the orthocentroidal circle with center $J = (2, 0)$ and radius 1 is

$$r^2 - 4r\cos\theta + 3 = 0. \tag{1}$$

This circle is enclosed by the curve defined by

$$r^2 - 2r(\cos\theta + 1) + 3 = 0. \tag{2}$$

Let Γ_1 denote the region enclosed by the closed curve defined by (2) for $\theta > 0$ and (1) for $\theta \leq 0$. Include the boundary when using (2) but exclude it when using (1). Delete the unique point Z in the interior which renders GJZ equilateral. (See Figure 1). Let Γ_2 be the reflection of Γ_1 in the Euler line. Let $\Gamma = \Gamma_1 \cup \Gamma_2$ so Γ consists of the set of points inside or on (2) for any θ, save that G and H are removed from the boundary and two points are deleted from the interior (the points Z such that GJZ is equilateral). It is easy to verify that if the points $(\sqrt{3}, \pm\pi/6)$ are restored to Γ, then it becomes convex, as do each of Γ_1 and Γ_2 if their deleted points are filled in.

2. The main theorem

Theorem. (a) *One Brocard point ranges freely over, and is confined to,* Γ_1*, and the other ranges freely over, and is confined to,* Γ_2.
(b) *The set of points which can be occupied by a Brocard point is* Γ.

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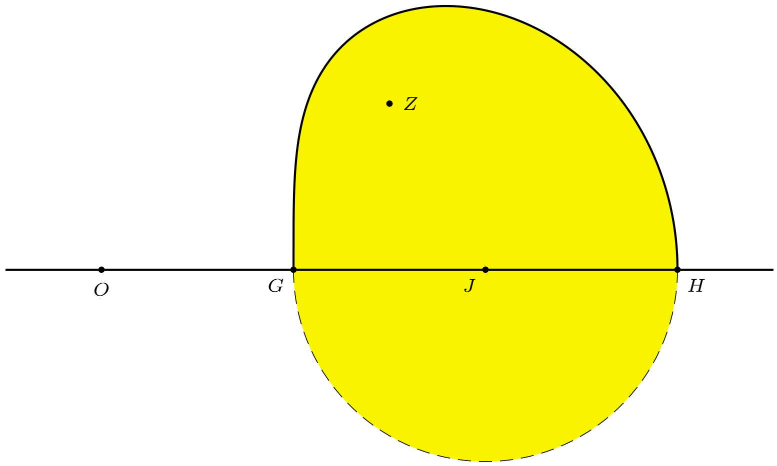

Figure 1

(c) *The points which can be inhabited by either Brocard point form the open orthocentroidal disk.*

(d) *The data consisting of O, G and one of the following items determines the sides of triangle ABC and which of the (generically) two possible orientations it has.*

(1) *the locations of the Brocard points without specifying which is which*;

(2) *the location of one Brocard point of specified type provided that it lies in the orthocentroidal disk*;

(3) *the location of one Brocard point of unspecified type outside the closed orthocentroidal disk together with the information that the other Brocard point lies on the same side or the other side of the Euler line*;

(4) *the location of a Brocard point of unspecified type on the orthocentroidal circle*.

Proof. First we gather some useful information. In [2] we established that

$$\frac{JK^2}{OG^2} = 1 - \frac{48[ABC]^2}{(a^2+b^2+c^2)^2} \tag{3}$$

where $[ABC]$ denotes the area of this triangle. It is well known [6] that

$$\cot\omega = \frac{(a^2+b^2+c^2)}{4[ABC]} \tag{4}$$

so

$$\frac{JK^2}{OG^2} = 1 - 3\tan^2\omega. \tag{5}$$

The sum of the powers of Ω and Ω' with respect to the orthocentroidal circle (with diameter GH) is 0. This result can be obtained by substituting the areal co-ordinates of these points into the areal equation of this circle [2].

$$J\Omega^2 + J\Omega'^2 = 2OG^2. \tag{6}$$

We can immediately conclude that $J\Omega$, $J\Omega' \leq OG\sqrt{2}$.

Now start to address the loci of the Brocard points. Observe that if we specify the locations of O, G and the symmedian point K (at a point within the orthocentroidal circle), then a triangle exists which gives rise to this configuration and its sides are determined [2]. In the subsequent discussion the points O and G will be fixed, and we will be able to conjure up triangles ABC with convenient properties by specifying the location of K. The angle α is just the directed angle $\angle KOG$ and ω can be read off from $JK^2/OG^2 = 1 - 3\tan\omega^2$.

We work with a non-equilateral triangle ABC. We adopt the convention, which seems to have majority support, that when standing at O and viewing K, the point Ω is diagonally to the left and Ω' diagonally to the right.

The Brocard or seven-point circle has diameter OK where K is the symmedian point, and the Brocard points are on this circle, and are mutual refections in the Brocard axis OK. It is well known that the Brocard angle manifests itself as

$$\omega = \angle KO\Omega = \angle KO\Omega'. \tag{7}$$

As the non-equilateral triangle ABC varies, we scale distances so that the length OG is 1 and we rotate as necessary so that the reference ray OG points in a fixed direction. Now let K be at an arbitrary point of the orthocentroidal disk with J deleted. Let $\angle KOJ = \alpha$, so $\angle JO\Omega' = \omega - \alpha$. Viewed as a directed angle the argument of Y in polar terms would be $\angle\Omega' OJ = \alpha - \omega$.

The positions of the Brocard points are determined by (5), (7) and the fact that they are on the Brocard circle. Let $r = O\Omega = O\Omega'$. By the cosine rule

$$J\Omega^2 = 4 + r^2 - 4r\cos(\omega + \alpha) \tag{8}$$

and

$$J\Omega'^2 = 4 + r^2 - 4r\cos(\omega - \alpha) \tag{9}$$

Now add equations (8) and (9) and use (6) so that

$$2OG^2 = 8 + 2r^2 - 8r\cos\omega\cos\alpha.$$

Recalling that the length OG is 1 we obtain

$$r^2 - 4r\cos\omega\cos\alpha + 3 = 0. \tag{10}$$

We focus on the Brocard point Ω with polar co-ordinates $(r, \alpha + \omega)$. The other Brocard point Ω' has co-ordinates $(r, \alpha - \omega)$, but reflection symmetry in the Euler line means that we need not study the region inhabited by Ω' separately.

Consider the possible locations of Ω for a specified $\alpha + \omega$. From (10) we see that its distance from the origin ranges over the interval

$$2\cos\omega\cos\alpha \pm \sqrt{4\cos^2\omega\cos^2\alpha - 3}.$$

This can be written

$$\cos(\alpha + \omega) + \cos(\alpha - \omega) \pm \sqrt{((\cos(\alpha + \omega) + \cos(\alpha - \omega))^2 - 3}. \tag{11}$$

Next suppose that $\alpha > 0$. This expression (11) is maximized when $\alpha = \omega$ and we use the plus sign. Since the product of the roots is 3, we see that the minimum distance also occurs when $\alpha = \omega$ and we use the minus sign.

Let $\theta = \angle \Omega OG$ for Ω on the boundary of the region under discussion, so $\theta = \omega + \alpha = 2\alpha = 2\omega$. Thus locus of the boundary when $\theta > 0$ is given by (10). Using standard trigonometric relations this transforms to

$$r^2 - 2r(\cos\theta + 1) + 3 = 0$$

for $\theta > 0$. This is equation (2). Note that it follows that Ω is on the boundary precisely when Ω' is on the Euler line because the argument of Ω' is $\alpha - \omega$.

Next suppose that $\alpha = 0$. Note that K cannot occupy J. When K is on the Euler line (so ABC is isosceles) equation (10) ensures that Ω is on the orthocentroidal circle. Also $0 < \omega < \pi/6$. Thus the points $(1, 0)$, $(\sqrt{3}, \pi/6)$ and $(3, 0)$ do not arise as possible locations for Ω. The endpoints of the GH interval are on the edge of our region. The more interesting exclusion is that of $(\sqrt{3}, \pi/6)$. We say that this is a *forbidden point* of Ω.

Now suppose that $\alpha < 0$. This time equation (11) tells another story. The expression is maximized (and minimized as before) when $\omega = 0$ which is illegal. In this region the boundary is not attained, and the point Ω' free to range on the axis side of the curve defined by

$$r^2 - 4r\cos\theta + 3 = 0$$

for $\theta = \alpha < 0$. Notice that this is the equation of the boundary of the orthocentroidal circle (1). The reflection of this last curve gives the unattained boundary of Ω' when $\theta > 0$, but it is easy to check that $r^2 - 2(\cos\theta + 1)r + 3 = 0$ encloses the relevant orthocentroidal semicircle and so is the envelope of the places which may be occupied by at least one Brocard point.

Moreover, our construction ensures that every point in Γ_1 arises a possible location for Ω.

We have proved (a). Then (b) follows by symmetry, and (c) is a formality.

Finally we address (d). Suppose that we are given O, G and a Brocard point X. Brocard points come in two flavours. If X is outside the open orthocentroidal disk on the side $\theta > 0$, then the Brocard point must be Ω, and if $\theta < 0$, then it must be Ω'. If X is in the disk, we need to be told which it is. Suppose without loss of generality we know the Brocard point is Ω. We know $J\Omega$ so from (6) we know $J\Omega'$. We also know $O\Omega' = O\Omega$, so by intersecting two circles we determine two candidates for the location of Ω'. Now, if Ω is outside the closed orthocentroidal disk then the two candidates for the location of Ω' are both inside the open orthocentroidal disk and we are stuck unless we know on which side of the Euler line Ω' can be found. If Ω is on the orthocentroidal circle then ABC is isosceles and the position of Ω' is known. If Ω is in the open orthocentroidal disk then only one of the two candidate positions for Ω' lies inside the set of points over which Ω' may range so the location of Ω' is determined. Now the Brocard circle and O are determined, so the antipodal point K to O is known. However, in [2] we showed that the triangle sides and its orientation may be recovered from O, G and K. We are done. □

In Figure 2, we illustrate the loci of the Brocard points for triangles with various Brocard angles ω. These are enveloped by the curve (2). The region Γ_1 is shaded.

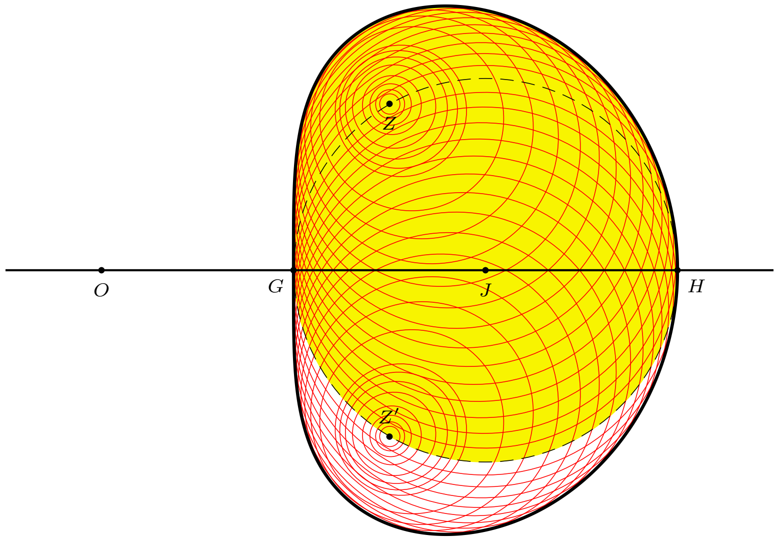

Figure 2

3. A qualitative description

We present an informal and loose qualitative description of the movements of Ω and Ω' as we steer K around the orthocentroidal disk. First consider K near G, with small positive argument. Both Brocard points are close to K, Ω just outside and Ω' just inside orthocentroidal circle. Now let K make one orbit, starting with positive arguments, just inside the circle. All the time ω stays small, and the two Brocard points nestle close to K in roughly the same configuration until K passes H at which point the Brocard points change roles; Ω' dives inside the circle and Ω moves outside. Though their paths cross, the Brocard points do not actually meet of course. For the second half of the passage of K just inside the circle it is Ω which is just inside the circle and Ω' which is just outside. When reaching the Euler line near G, the Brocard points park symmetrically on the circle with Ω having positive argument.

Now move K along the Euler line towards J; the Brocard points move round the circle, mutual reflections in the Euler line and Ω has positive argument. Triangle ABC is isosceles. As K approaches J each Brocard point approaches its forbidden point. Let K make a small swerve round J to rejoin the Euler line on the other side. Suppose that the swerve is on the side $\theta > 0$. In this case Ω swerves round its forbidden point outside the circle, and Ω' swerves inside, both points rejoining the circle almost immediately. Now K sails along the Euler line and the three points come close again together as K approaches H.

Next let K be at an arbitrary legal position on the Euler line. We fix OK and increase the argument of K. Both Brocard points also move in the same general direction; Ω leaves the orthocentroidal disk and heads towards the boundary, and

Ω' chases K. When K reaches a certain critical point, Ω reaches the boundary and at the same instant, Ω' crosses the Euler line. Now K keeps moving towards Ω followed by Ω', but Ω reverses direction and plunges back towards K. The three points come close together as K approaches the unattainable circle boundary. The process will unwrap as K reverses direction until it arrives back on the Euler line.

Another interesting tour which K can take is to move with positive arguments and starting near G along the path defined by the following equation:

$$r^2 - 4r\cos\theta + 3 + 3\tan^2\theta = 0 \qquad (12)$$

This is the path of critical values which has Ω moving on the boundary of Γ_1 and Ω' on the Euler line. Now $S\Omega'O$ is a right angle so in this particular sweep the position of Ω' is the perpendicular projection on the Euler line of the position of K. We derive (12) as follows. Take (2) and express θ in terms of $\theta/2$ and multiply through by $\sec^2\frac{\theta}{2}$. Now $OK = r\sec\frac{\theta}{2}$. Relabel by replacing $r\sec\frac{\theta}{2}$ by r and then replacing the remaining occurence of $\frac{\theta}{2}$ by θ.

A final journey of note for K is obtained by fixing the Brocard angle ω. Then K is free to range over a circle with center J and radius KJ where $KJ^2 = 1 - 3\tan^2\omega$ because of (3). The direct similarity type of triangles $OK\Omega$ and $OK\Omega'$ will not change, so Ω and Ω' will each move round circles. As K takes this circular tour through the moduli space of directed similarity types of triangle, we make the same journey through triangles as when a triangle vertex takes a trip round a Neuberg circle.

4. Discussion

We can obtain an areal equation of the boundary of Γ using the fact that one Brocard point is on the boundary of Γ exactly when the other is on the Euler line. The description is therefore the union of two curves, but the points G and H must be removed by special fiat.

The equation of the Euler line is

$$(b^2+c^2-a^2)(b^2-c^2)x+(c^2+a^2-b^2)(c^2-a^2)y+(a^2+b^2-c^2)((a^2-c^2)z = 0.$$

The Brocard point (a^2b^2, b^2c^2, c^2a^2) lies on the Euler line if and only if

$$a^6c^2 + b^6a^2 + c^6b^2 = a^4b^4 + b^4c^4 + c^4a^4.$$

The locus of the other Brocard point $x = c^2a^2, y = a^2b^2, z = b^2c^2$ is then given by

$$x^3y^2 + y^3z^2 + z^3x^2 = xyz(x^2 + y^2 + z^2).$$

To get the other half of the boundary we must exchange the roles of Ω and Ω' and this yields

$$x^3z^2 + y^3x^2 + z^3y^2 = xyz(x^2 + y^2 + z^2)$$

so the locus is a quintic in areal co-ordinates.

Equation (10) exhibits an intriguing symmetry between $\alpha = \angle KOG$ and $\omega = \angle\Omega OS$ which we will now explain. Suppose that we are given the location Y of a Brocard point within the orthocentroidal circle, but not the information as to whether the Brocard point is Ω or Ω'. If this Brocard point is Ω, we call the location of the other Brocard point Ω_1 and the corresponding symmedian point K_1. On the other hand if the Brocard point at Y is Ω' we call the location of the other Brocard point Ω_2 and the corresponding symmedian point K_2. Let the respective Brocard angles be ω_1 and ω_2.

We have two Brocard circles, so $\angle K_2YO = \angle K_1YO = \pi/2$ and therefore Y lies on the line segment K_1K_2. Using equation (6) we conclude that the lengths $J\Omega_1$ and $J\Omega_2$ are equal. Also $O\Omega_1 = OY = O\Omega_2$. Therefore Ω_1 and Ω_2 are mutual reflections in the Euler line. Now

$$\angle\Omega_2O\Omega_1 = \angle\Omega_2OG + \angle GO\Omega_1 = 2\omega_1 + 2\omega_2.$$

However, the Euler line is the bisector of $\angle\Omega_2OG$ so $\angle K_2OG = \omega_1$ and $\angle GOK_1 = \omega_2$. Thus in the "ω, α" description of Ω_2 which follows from equation (10), we have $\omega = \omega_2$ and $\alpha = \omega_1$. However, exchanging the roles of left and right in the whole discussion (the way in which we have discriminated between the first and second Brocard points), the resulting description of Ω_1 would have $\omega = \omega_1$ and $\alpha = \omega_2$. The symmetry in (10) is explained.

There are a pair of triangles determines by by the quadruple

$$(O, G, \Omega, \Omega')$$

using the values (O, G, Ω_2, Y) and (O, G, Y, Ω_1) which are linked via their common Brocard point in the orthocentroidal disk. We anticipate that there may be interesting geometrical relationships between these pairs of non-isosceles triangles.

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