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Proof by Picture: Products and Reciprocals of Diagonal Length Ratios in the Regular Polygon

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Abstract. These "proofs by picture" link the geometry of the regular n-gon to formulae concerning the arithmetic of real cyclotomic fields. We illustrate the formula for the product of diagonal length ratios

$$r_h r_k = \sum_{i=1}^{\min[k,h,n-k,n-h]} r_{|k-h|+2i-1}.$$

and that for the reciprocal of diagonal length ratios when $\gcd(n,k)=1$,

$$\frac{1}{r_k} = \sum_{j=1}^{s} r_{k(2j-1)}, \qquad \text{where } s = \min\{j>0 : kj \equiv \pm 1 \bmod n\}.$$

1. Introduction

Consider a regular n-gon. Number the diagonals $d_1, d_2, \ldots, d_{n-1}$ (as shown in Figure 1.1 for $n=9$) including the sides of the polygon as d_1 and d_{n-1}. Although the length of d_i equals that of d_{n-i}, we shall use all $n-1$ subscripts since this simplifies our formulae concerning the diagonal lengths.

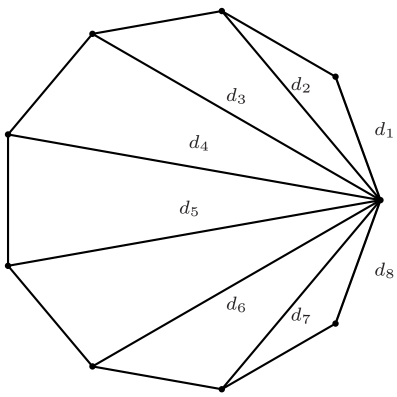

Figure 1.1

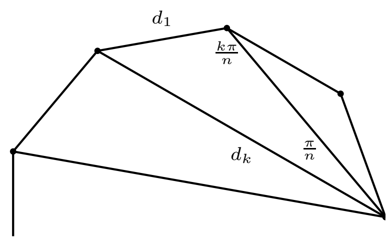

Figure 1.2

Ratios of the lengths of the diagonals are given by the law of sines. The ratio of the length of d_k to that of d_j is $\dfrac{\sin\frac{k\pi}{n}}{\sin\frac{j\pi}{n}}$. In particular, Figure 1.2 shows us that the

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ratio of the length of d_k to that of d_1 is $\dfrac{\sin\frac{k\pi}{n}}{\sin\frac{\pi}{n}}$. This ratio of sines will be denoted r_k. Note that if the length of the side $d_1 = 1$, r_k is simply the length of the k-th diagonal.

2. Products of diagonal length ratios

It is an exercise in the algebra of cyclotomic polynomials to show that

$$r_h r_k = \sum_{i=1}^{\min[k,h,n-k,n-h]} r_{|k-h|+2i-1}.$$

This formula appears in Steinbach [1] for the case where $h + k \le n$. Steinbach names it the *diagonal product formula* and makes use of it to derive a number of interesting properties of the diagonal lengths of a regular polygon. It is not hard to extend the formula to cover all $n-1$ values of h and k.

In order to understand the geometry of the *diagonal product formula*, consider two regular n-gons with the side of the larger equal to some diagonal of the smaller. Denote the diagonal lengths by $\{s_i\}_{i=1,\,\ldots,n-1}$ for the smaller polygon and $\{l_i\}_{i=1,\,\ldots,n-1}$ for the larger, so that $l_1 = s_k$ for some k. In this case, since $r_k = \frac{s_k}{s_1} = \frac{l_1}{s_1}$ and $r_h = \frac{l_h}{l_1}$, the product $r_k r_h$ becomes $\frac{l_h}{s_1}$ and when we multiply through by s_1 the diagonal product formula becomes

$$l_h = \sum_{i=1}^{\min[k,h,n-k,n-h]} s_{|k-h|+2i-1}.$$

In other words, each of the larger diagonal lengths is expressible as a sum of the smaller ones.

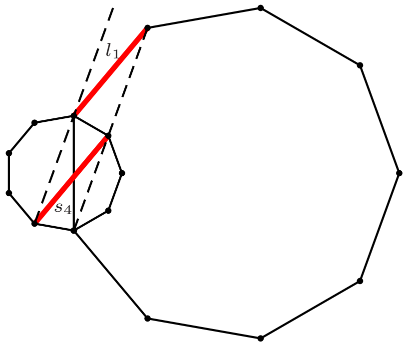

Figure 2.1

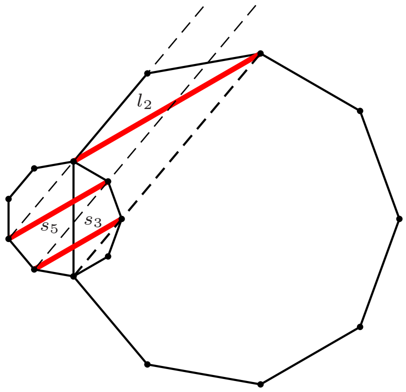

Figure 2.2

Figures 2.1-2.4 illustrate what happens when the nonagon is enlarged so that $l_1 = s_4$. The summation formula for the diagonals can be visualized by projecting the left edge of the larger polygon onto each of its other edges in turn. We observe from the first pair of nonagons that $l_1 = s_4 s_1 = s_4$, from the second that $l_2 =$

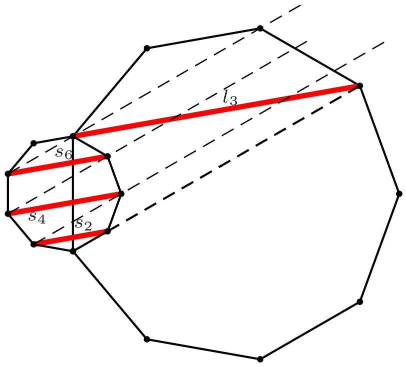

Figure 2.3

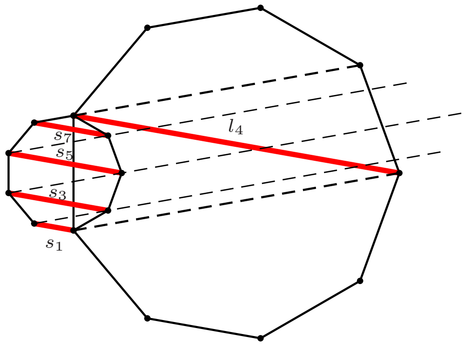

Figure 2.4

$s_4s_2 = s_3 + s_5$, from the third that $l_3 = s_4s_3 = s_2 + s_4 + s_6$, and from the last that $l_4 = s_4s_4 = s_1 + s_3 + s_5 + s_7$. This is exactly what the *diagonal product formula* predicts.

When n and k are both even, the polygons do not have the same orientation, but the same strategy of projecting onto each side of the larger polygon in turn still works. Figure 3 shows the case $(n, k) = (6, 2)$. The sums are

$$\begin{aligned} l_1 =& s_2s_1 = s_2, \\ l_2 =& s_2s_2 = s_1 + s_3, \\ l_3 =& s_2s_3 = s_3 + s_4. \end{aligned}$$

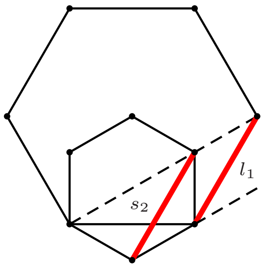

Figure 3.1

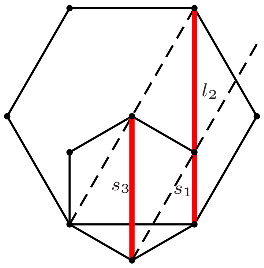

Figure 3.2

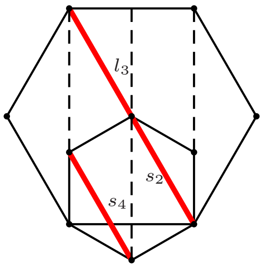

Figure 3.3

3. Reciprocal of diagonal length ratios

Provided $\gcd(k, n) = 1$, the diagonal length ratio r_k is a unit in the ring of integers of the real subfield of $\mathbf{Q}[\xi]$, ξ a primitive n-th root of unity. The set of ratios $\{r_i\}$ with $\gcd(i, n) = 1$ and $i \leq \frac{n}{2}$ forms a basis for this field [2]. Knowing that this was the case, we searched for a formula to express $\frac{1}{r_k}$ as an integral linear combination of diagonal length ratios. This time we found the picture first. We assume that the polygon has unit side, so that $r_i =$ the length of the i-th diagonal.

First note that $\frac{1}{r_k}$ is equal to the length of the line segment obtained when diagonal k intersects diagonal 2 as shown in Figure 4.1 for $(n, k) = (8, 3)$. In order to express this length in terms of the r_i, $i = 1, \ldots, n-1$, notice that, as in Figure 4.2 for $(n, k) = (7, 2)$, one can set off along diagonal k and zigzag back and forth, alternately parallel to diagonal k and in the vertical direction, until one arrives at a vertex adjoining the starting point. Summing the lengths of the diagonals parallel to diagonal k, with positive or negative sign according to the direction of travel, will give the desired reciprocal. For example follow the diagonals shown in Figure 4.2 to see that $\frac{1}{r_2} = r_2 + r_6 - r_4$.

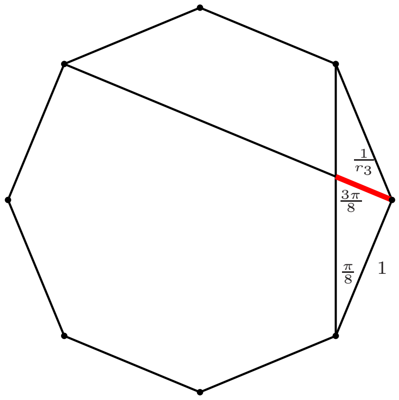

Figure 4.1

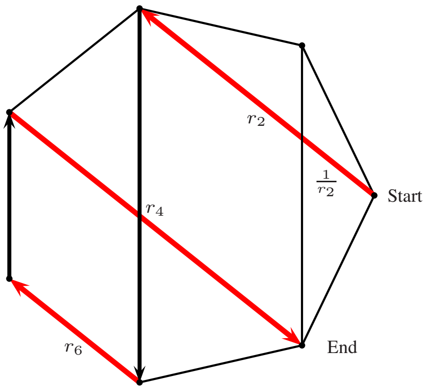

Figure 4.2

Following the procedure outlined above, the head of each directed diagonal used in the sum is k vertices farther around the polygon from the previous one. Also as we zigzag our way through the polygon, the positive contributions to the sum occur as we move in one direction with respect to diagonal k, while the negative contributions occur as we move the other way. In fact, allowing $r_i = \frac{\sin\frac{i\pi}{n}}{\sin\frac{\pi}{n}}$ to be defined for $i > n$, we realized that $r_i = -r_{2n-i}$ would have the correct sign to produce the simplest formula for the reciprocal, which is

$$\frac{1}{r_k} = \sum_{j=1}^{s} r_{k(2j-1)}, \qquad \text{where } s = \min\{j > 0 : kj \equiv \pm 1 \bmod n\}.$$

Once we had discovered this formula, we found it that it was a messy but routine exercise in cyclotomic polynomial algebra to verify its truth.

Although we are almost certain that these formulas for manipulating the diagonal length ratios must be in the classical literature, we have not been able to locate them, and would appreciate any lead in this regard.

References

[1] P. Steinbach, Golden fields: a case for the heptagon, *Math. Mag.*, 70 (1997) 22–31.
[2] L. C. Washington, *Introducion to Cyclotomic Fields*, 2nd edition, Springer-Verlag, New York, 1982.

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