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Grassmann cubics and Desmic Structures

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Abstract. We show that each cubic of type $\mathrm{n}\mathcal{K}$ which is not of type $\mathrm{c}\mathcal{K}$ can be described as a Grassmann cubic. The geometry associates with each such cubic a cubic of type $\mathrm{p}\mathcal{K}$. We call this the parent cubic. On the other hand, each cubic of type $\mathrm{p}\mathcal{K}$ has infinitely many child cubics. The key is the existence of a desmic structure associated with parent and child. This extends work of Wolk by showing that, not only do (some) points of a desmic structure lie on a cubic, but also that they actually generate the cubic as a locus. Along the way, we meet many familiar cubics.

1. Introduction

In Hyacinthos, #3991 and follow-up, Ehrmann and others gave a geometrical description of cubics as loci. They showed that each cubic of type $\mathrm{n}\mathcal{K}_0$ has an associated sister of type $\mathrm{p}\mathcal{K}$, and that each cubic of type $\mathrm{p}\mathcal{K}$ has three sisters of type $\mathrm{n}\mathcal{K}_0$. Here, we show that each cubic of type $\mathrm{n}\mathcal{K}$ but not of type $\mathrm{c}\mathcal{K}$ has a parent of type $\mathrm{p}\mathcal{K}$, and every cubic of type $\mathrm{p}\mathcal{K}$ has infinitely many children, but just three of type $\mathrm{n}\mathcal{K}_0$. Our results do not appear to extend to cubics of type $\mathrm{c}\mathcal{K}$, so the geometry must be rather different. Throughout, we use barycentric coordinates. We write the coordinates of P as $p : q : r$. We are interested in isocubics, that is circumcubics which are invariant under an isoconjugation. The theory of isocubics is beautifully presented in [1]. There we learn that an isocubic has an equation of one of the following forms :

$$\begin{array}{ll} \mathrm{p}\mathcal{K}(W,R): & rx(wy^2 - vz^2) + sy(uz^2 - wx^2) + tz(vx^2 - uy^2) = 0. \\ \mathrm{n}\mathcal{K}(W,R,k): & rx(wy^2 + vz^2) + sy(uz^2 + wx^2) + tz(vx^2 + uy^2) + kxyz = 0. \end{array}$$

The point $R = r : s : t$ is known as the pivot of the cubic $\mathrm{p}\mathcal{K}$, and the root of the cubic $\mathrm{n}\mathcal{K}$ $W = u : v : w$ is the pole of the isoconjugation. This means that the isoconjuagte of $X = x : y : z$ is $\frac{u}{x} : \frac{v}{y} : \frac{w}{z}$. We may view W as the image of G under the isoconjugation. The constant k in the latter equation is determined by a point on the curve which is not on a sideline. Another interpretation of k appears below. One important subclass of type $\mathrm{n}\mathcal{K}$ occurs when $k = 0$. We have

$$\mathrm{n}\mathcal{K}_0(W,R): \qquad rx(wy^2 + vz^2) + sy(uz^2 + wx^2) + tz(vx^2 + uy^2) = 0.$$

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Another subclass consists of the conico-pivotal isocubics. These are defined in terms of the root R and node $F = f : g : h$. The equation has the form

$$\mathrm{c}\mathcal{K}(\#F, R): \qquad rx(hy - gz)^2 + sy(fz - hx)^2 + tz(gx - fy)^2 = 0.$$

It is the $\mathrm{n}\mathcal{K}(W, R, k)$ with $W = f^2 : g^2 : h^2$, so F is a fixed point of the isoconjugation, and $k = -2(rgh + shf + tfg)$. We require a further concept from [1], that of the polar conic of R for $\mathrm{p}\mathcal{K}(W, R)$. In our notation, this has equation

$$\mathrm{p}\mathcal{C}(W, R): \qquad (vt^2 - ws^2)x^2 + (wr^2 - ut^2)y^2 + (us^2 - vr^2)z^2 = 0.$$

Notice that, $\mathrm{p}\mathcal{C}(W, R)$ contains the four fixed points of the isoconjugation, so is determined by one other point, such as R. Hence, as befits a polar object, if S is on $\mathrm{p}\mathcal{C}(W, R)$, then R is on $\mathrm{p}\mathcal{C}(W, S)$. Also, for fixed W, the class of cubics of type $\mathrm{p}\mathcal{K}$ splits into disjoint subclasses, each subclass having a common polar conic. We also remark that the equation for the polar conic vanishes identically when W is the barycentric square of R. In such a case, it is convenient to regard the class of $\mathrm{p}\mathcal{K}(W, R)$ as consisting of just $\mathrm{p}\mathcal{K}(W, R)$ itself.

Definition 1. (a) *For a point X with barycentrics $x : y : z$, the A-harmonic of X is the point X_A with barycentrics $-x : y : z$. The B- and C-harmonics, X_B and X_C, are defined analagously.*
(b) *The harmonic associate of the triangle $\triangle PQR$ is the triangle $\triangle P_A Q_B R_C$.*

Observe that $\triangle X_A X_B X_C$ is the anticevian triangle of X. Also, if X is on $\mathrm{p}\mathcal{C}(W, R)$ then X_A, X_B, X_C also lie on $\mathrm{p}\mathcal{C}(W, R)$. In the preamble to $X(2081)$ in [5] we meet Gibert's PK- and NK-transforms. These are defined in terms of isogonal conjugation. We shall need the general case for W-isoconjugation.

Definition 2. *Suppose that W is a fixed point, let P^* denote the W-isoconjugate of P.*
(a) *The PKW-transform of P is PK(W,P), the intersection of the tripolars of P and P^* (if $P \neq P^*$).*
(b) *The NKW-transform of P is $NK(W, P)$, the crosspoint of P and P^*.*

Note that $PK(W, P)$ is the perspector of the circumconic through P and P^*. $PK(W, P)$ occurs several times in our work, in apparently unrelated contexts. From these definitions, if $W = u : v : w$ and $P = p : q : r$, then

$$\begin{aligned} PK(W, P) &= pu(r^2v - q^2w) : qv(p^2w - r^2u) : rw(q^2u - p^2v), \\ NK(W, P) &= pu(r^2v + q^2w) : qv(p^2w + r^2u) : rw(q^2u + p^2v). \end{aligned}$$

Note that $PK(W, R)$ is undefined if $W = R^2$.

2. Grassmann cubics associated with desmic Structures

Definition 3. *Let $\Delta = \triangle ABC$ be the reference triangle. Let $\Delta' = \triangle A'B'C'$ where A' is not on BC, B' is not on CA and C' is not on AB.*
(a) $GP(\Delta') = \{P :$ *the triangle with vertices $A'P \cap BC$, $B'P \cap CA$, $C'P \cap AB$ is perspective with Δ*$\}$,
(b) $GN(\Delta') = \{P :\ A'P \cap BC,\ B'P \cap CA,\ C'P \cap AB$ *are collinear*$\}$.

We note that $GN(\Delta)$ is a special case of a Grassmann cubic. There are three examples in the current (November 2004) edition of [2]. The Darboux cubic $\mathrm{p}\mathcal{K}(K, X(20))$ is $K004$ in Gibert's catalogue. Under the properties listed is the fact that it is the locus of points whose pedal triangle is a cevian triangle. This is $GP(\triangle A'B'C')$, where the vertices are the infinite points on the altitudes. The corresponding GN is then the union of the line at infinity and the circumcircle, a degenerate cubic. In the discussion of the cubic $\mathrm{n}\mathcal{K}(K, X(5))$ - Gibert's $K216$ - it is mentioned that it is GN for a certain triangle, and that the corresponding GP is the Neuberg cubic, $\mathrm{p}\mathcal{K}(K, X(30))$ - Gibert's $K001$. As we shall see, there are other examples where suitable triangles are listed, and others where they can be identified.

Lemma 1. *If $\Delta' = \triangle A'B'C'$ has $A' = a_1 : a_2 : a_3$, $B' = b_1 : b_2 : b_3$, $C' = c_1 : c_2 : c_3$, then,*
(a) *$GP(\Delta')$ has equation*

$$(a_2x - a_1y)(b_3y - b_2z)(c_1z - c_3x) + (a_1z - a_3x)(b_1y - b_2x)(c_2z - c_3y) = 0,$$

(b) *$GN(\Delta')$ has equation*

$$(a_2x - a_1y)(b_3y - b_2z)(c_1z - c_3x) - (a_1z - a_3x)(b_1y - b_2x)(c_2z - c_3y) = 0.$$

(c) *Each locus contains A, B, C and A', B', C'.*

Proof. It is easy to see that, if $P = x : y : z$, then

$$\begin{aligned} A_P &= A'P \cap BC = 0 : a_2x - a_1y : a_3x - a_1z, \\ B_P &= B'P \cap CA = b_1y - b_2x : 0 : b_3y - b_2z, \\ C_P &= C'P \cap AB = c_1z - c_3x : c_2z - c_3y : 0. \end{aligned}$$

The condition in $GN(\Delta)$ is equivalent to the vanishing of the determinant of the coefficients of these points. This gives (b).

The condition in $GP(\Delta)$ is equivalent to the concurrence of AA_P, BB_P and CC_P. This is another determinant condition. The determinant is formed from the previous one by changing the sign of one entry in each row. This gives (a). Once we have the equations, (c) is clear. □

We will also require a condition for a triangle $\triangle A'B'C'$ to be perspective with $\triangle ABC$.

Lemma 2. *If $\Delta = \triangle A'B'C'$ has $A' = a_1 : a_2 : a_3$, $B' = b_1 : b_2 : b_3$, $C' = c_1 : c_2 : c_3$, then,*
(a) *$\triangle A'B'C'$ is perspective with $\triangle ABC$ if and only if $a_2b_3c_1 = a_3b_1c_2$.*
(b) *$\triangle A'B'C'$ is triply perspective with $\triangle ABC$ if and only if $a_1b_2c_3 = a_2b_3c_1 = a_3b_1c_2$.*

Proof. We observe that the perspectivity in (a) is equivalent to the concurrence of AA', BB' and CC'. The given equality expresses the condition for the intersection of AA' and BB' to lie on CC'. Part (b) follows by noting that $\triangle A'B'C'$ is triply perspective with $\triangle ABC$ if and only if each of $\triangle A'B'C'$, $\triangle B'C'A'$ and $\triangle C'A'B'$ is perspective with $\triangle ABC$. □

We observe that each of the equations in Lemma 1 has the form

$$x(f_1y^2 + f_2z^2) + y(g_1z^2 + g_2x^2) + z(h_1x^2 + h_2y^2) + kxyz = 0. \qquad (1)$$

This has the correct form for the cubic to be of type $\mathrm{p}\mathcal{K}$ or $\mathrm{n}\mathcal{K}$.

Theorem 3. *For a triangle $\Delta' = \triangle A'B'C'$,*
(a) *$GP(\Delta)$ is of type $\mathrm{p}\mathcal{K}$ if and only if Δ' is perspective with $\triangle ABC$.*
(b) *If Δ' is degenerate, then $GN(\Delta')$ is degenerate.*
Suppose that $GN(\Delta')$ is non-degenerate. Then
(c) *$GN(\Delta')$ is of type $\mathrm{n}\mathcal{K}$ if and only if Δ' is perspective with $\triangle ABC$.*
(d) *$GN(\Delta')$ is of type $\mathrm{n}\mathcal{K}_0$ if and only if Δ' is triply perspective with $\triangle ABC$.*

Proof. Suppose that $A' = a_1 : a_2 : a_3$, $B' = b_1 : b_2 : b_3$, $C' = c_1 : c_2 : c_3$, with $a_1b_2c_3 \neq 0$.

(a) We begin by observing that equation (1) gives a cubic of type $\mathrm{p}\mathcal{K}$ if and only if $f_1g_1f_1 + f_2g_2h_2 = 0$ and $k = 0$. The equation in Lemma 1(a) has $k = 0$ if and only if $a_2b_3c_1 = a_3b_1c_2$. By Lemma 2(a), this is the condition for the triangles to be perspective. A Maple calculation shows that the other condition for a $\mathrm{p}\mathcal{K}$ is satisfied if the triangles are perspective. This establishes (a).

(b) If Δ' is degenerate, then A', B' and C' lie on a line $\mathcal{L}$. For any P on $\mathcal{L}$, the intersection of PA' with BC lies on $\mathcal{L}$, as do those of PB' with CA, and PC' with AB. Thus, these intersections are collinear, so P is on $GN(\Delta')$. Now the locus contains the line $\mathcal{L}$, so that it must be degenerate.

Now suppose that the locus $GN(\Delta')$ is non-degenerate.

(c) Equation (1) gives a cubic of type $\mathrm{n}\mathcal{K}$ if and only if $f_1g_1f_1 - f_2g_2h_2 = 0$. The equation in Lemma 1(b) has this property if and only if

$$(a_2b_3c_1 - a_3b_1c_2)D = 0,$$

where D is the determinant of the matrix whose rows are the coordinates of A', B' and C'. Now, $D = 0$ if and only if A', B' and C' are collinear, so Δ' and hence $GN(\Delta')$ are degenerate. By Lemma 2(a), the other condition is equivalent to the perspectivity of the triangles.

(d) For a cubic of type $\mathrm{n}\mathcal{K}_0$, we require $a_2b_3c_1 - a_3b_1c_2 = 0$, as in (c). We also require that $k = 0$. From the equation in Lemma 1(b), $k = a_2b_3c_1 + a_3b_1c_2 - 2a_1b_2c_3$. These two conditions are equivalent to triple perspectivity by Lemma 2(b). $\square$

Our work so far leads us to consider triangles $\triangle A'B'C'$ perspective to $\triangle ABC$, with A' not on BC, B' not on CA, and C' not on AB. Looking at our loci, we are led to consider a further triangle, also perspective with $\triangle ABC$. This turns out to be the desmic mate of $\triangle A'B'C'$, so we are led to consider desmic structures which include the points A, B and C.

Theorem 4. *Suppose that $\triangle A'B'C'$ is perspective to $\triangle ABC$, with A' not on BC, B' not on CA, and C' not on AB. Let the perspector be $P_1 = p_{11} : p_{12} : p_{13}$.*

(a) *Suitably normalized, we have*

$$\begin{aligned} A' &= p_{21} : p_{12} : p_{13}, \\ B' &= p_{11} : p_{22} : p_{13}, \\ C' &= p_{11} : p_{12} : p_{23}. \end{aligned}$$

Let

$$\begin{aligned} W &= p_{11}p_{21} : p_{12}p_{22} : p_{13}p_{23}, \\ R &= p_{11} - p_{21} : p_{12} - p_{22} : p_{13} - p_{23}, \\ S &= p_{11} + p_{21} : p_{12} + p_{22} : p_{13} + p_{23}. \end{aligned}$$

(b) $GP(\triangle A'B'C') = \mathrm{p}\mathcal{K}(W, S)$,
(c) $GN(\triangle A'B'C') = \mathrm{n}\mathcal{K}(W, R, 2(p_{21}p_{22}p_{23} - p_{11}p_{12}p_{13}))$.

Let $A'' = p_{11} : p_{22} : p_{23}$, $B'' = p_{21} : p_{12} : p_{23}$, $C'' = p_{21} : p_{22} : p_{13}$. *These are the* W*-isoconjugates of* A', B' *and* C'.
(d) $\triangle A''B''C''$ *is perspective with* $\triangle ABC$*, with perspector* $P_2 = p_{21} : p_{22} : p_{23}$.
(e) $\triangle A''B''C''$ *is perspective with* $\triangle A'B'C'$*, with perspector* S.
(f) (P_1, P_2, R, S) *is a harmonic range.*
(g) $GP(\triangle A''B''C'') = GP(\triangle A'B'C')$. *The common locus includes* P_1, P_2 *and* S.
(h) $GN(\triangle A''B''C'') = GN(\triangle A'B'C')$. *The common locus includes the intersections of the tripolar of* R *with the sidelines.*
(i) $\triangle ABC$, $\triangle A'B'C'$ *and* $\triangle A''B''C''$ *have common perspectrix, the tripolar of* R.
(j) P_1 *and* P_2 *lie on* $\mathrm{p}\mathcal{K}(W, R)$.

Proof. (a) Since we have the perspector, the coordinates of the vertices A', B', C' must be as described.

(b),(c) These are simply verifications using the equations in Lemma 1.

(d) This follows at once from the coordinates of A'', B'', C''.

(e) This requires the calculations that the lines $A'A''$, $B'B''$ and $C'C''$ all pass through S.

(f) The coordinates of the points make this clear.

(g),(h) First, we note that, if we interchange the roles of P_1 and P_2, we get the same equations. The fact that the given points lie on the respective loci are simply verifications.

(i) Once again, this can be checked by calculation. We can also argue geometrically. Suppose that a point X on $B'C'$ lies on the locus. Then XB' and XC' are $B'C'$, so this must be the common line. Then XA' must meet BC on this line. But X is on $B'C'$, so $X = BC \cap B'C'$. If X also lies on $B''C''$, then $X = BC \cap B''C''$. This shows that we must have a common perspectrix. The identification of the perspectrix uses the fact that the cubic meets the each sideline of $\triangle ABC$ in just three points, two vertices and the intersection with the given tripolar.

(j) This is a routine verification. □

In the notation of Theorem 4, we have a desmic structure with the twelve points described as vertices A, B, C, A', B', C', A'', B'', C'', perspectors P_1, P_2, S.

Many authors describe S as the desmon, and R as the harmon of the structure. Some refer to P_1 as the perspector, and P_2 as the coperspector. This description of a desmic structure with vertices including A, B, C is discussed by Barry Wolk in Hyacinthos #462. He observed that the twelve points all lie on $\mathrm{p}\mathcal{K}(W, S)$. What may be new is the fact that the other vertices may be used to generate this cubic as a locus, and the corresponding $\mathrm{n}\mathcal{K}$ as a Grassmann cubic.

Notice that the desmic structure in not determined by its perspectors. If we choose barycentrics for P_1, we need to scale P_2 so that the barycentrics of the desmon and harmon are, respectively, the sum and difference of those of P_1 and P_2. When P_2 is suitably scaled, we say that the perspectors are normalized. The normalization is determined by a single vertex, provided neither perspector is on a sideline.

There are two obvious questions.

(1) Is every cubic of type $\mathrm{p}\mathcal{K}$ a locus of type GP associated with a desmic structure?

(2) Is every cubic of type $\mathrm{n}\mathcal{K}$ a locus of type GN associated with a desmic structure?

The answer to (1) is that, with six exceptions, each point on a $\mathrm{p}\mathcal{K}$ is a perspector of a suitable desmic structure. The answer to (2) is more complicated. There is a class of $\mathrm{n}\mathcal{K}$ which do not possess a suitable desmic structure. This is the class of conico-pivotal isocubics. For each other cubic of type $\mathrm{n}\mathcal{K}$, there is a unique desmic structure.

Theorem 5. *Suppose that P is a point on $\mathrm{p}\mathcal{K}(W, S)$ which is not fixed by W-isoconjugation, and is not S or its W-isoconjugate. Then there is a unique desmic structure with vertices A, B, C and perspector $P_1 = P$ with locus $GP = \mathrm{p}\mathcal{K}(W, S)$.*

Proof. For brevity, we shall write X^* for the W-isoconjugate of a point X. As P is on $\mathrm{p}\mathcal{K}(W, S)$, S is on PP^*. Then $S = mP + nP^*$, for some constants m, n, with $mn \neq 0$. If $P = p : q : r$, $W = u : v : w$, put $A' = nu/p : mq : mr$, $B' = mp : nv/q : mr$, $C' = mp : mq : nw/r$. From Theorem 4, this has locus $GP(\triangle A'B'C') = \mathrm{p}\mathcal{K}(W, S)$. □

Note that the conditions on P are necessary to ensure that S can be expressed in the stated from. For the second question, we proceed in two stages. First, we show that a cubic of type $\mathrm{n}\mathcal{K}$ has at most one suitable desmic structure. This identifies the vertices of the structure. We then show that this choice does lead to a description of the cubic as a Grassmann cubic. The first result uses the idea of A-harmonics introduced in the Introduction.

Theorem 6. *Throughout, we use the notation of Theorem 4.*
(a) *The vertices of the desmic structure on $\mathrm{n}\mathcal{K}(W, R, k)$ are A, B, C and intersections of $\mathrm{n}\mathcal{K}(W, R, k)$ with the cubics $\mathrm{p}\mathcal{K}(W, R_A)$, $\mathrm{p}\mathcal{K}(W, R_B)$, $\mathrm{p}\mathcal{K}(W, R_C)$.*
(b) *$\mathrm{n}\mathcal{K}(W, R, k)$ and $\mathrm{p}\mathcal{K}(W, R_A)$ touch at B and C, intersect at A, at the intersections of the tripolar of R with AB and AC, and at two further points.*

(c) *If* $W = f^2 : g^2 : h^2$*, then* $\mathrm{c}\mathcal{K}(\#F, R)$ *and* $\mathrm{p}\mathcal{K}(W, R_A)$ *touch at* B *and* C*, and meet at* A*, at the intersections of the tripolar of* R *with* AB *and* AC*, and twice at* F*.*
(d) *If the final two points in* (b) *coincide, then* $\mathrm{n}\mathcal{K}(W, R, k) = \mathrm{c}\mathcal{K}(\#F, R)$*, where* F *is such that* $W = F^2$*.*
(e) *If either of the final points in* (b) *lie on a sideline, then* $k = 2ust/r$*,* $2vrt/s$ *or* $2wrs/t$*, where* $W = u : v : w$*,* $R = r : s : t$*.*

Proof. We observe that the vertices of the desmic structure can be derived from the normalized versions of the perspectors P_1 and P_2. The normalization is such that $R = P_1 - P_2$ and $S = P_1 + P_2$. Now consider the loci derived from the perspectors P_1 and $-P_2$. From Theorem 4, the cubics are $\mathrm{p}\mathcal{K}(W, R)$ and $\mathrm{n}\mathcal{K}(W, S, k')$, for some k'. The point $A' = p_{21} : p_{12} : p_{13}$ is on $\mathrm{n}\mathcal{K}(W, R, k)$, so that $A'_A = -p_{21} : p_{12} : p_{13}$ is on $\mathrm{p}\mathcal{K}(W, R)$. Then A' is on $\mathrm{p}\mathcal{K}(W, R_A)$ as only the first term is affected by the sign change, and this involves the product of the first coordinates. Thus (a) holds.

Part (b) is largely computational. Obviously the cubics meet at A, B and C. The tangents at B and C coincide. The intersections with the sidelines in each case include the stated meetings with the tripolar of R. Since two cubics have a total of nine meets, there are two unaccounted for. These are clearly W-isoconjugate.

For part (c), we use the result of (b) to get seven intersections. Then we need only verify that the cubics meet twice at F. Now F is clearly on $\mathrm{p}\mathcal{K}(W, R_A)$. But F is a double point on $\mathrm{c}\mathcal{K}(\#F, R)$, so there are two intersections here.

Part (d) relies on a Maple calculation. Solving the equations for $\mathrm{n}\mathcal{K}(W, R, k)$ and $\mathrm{p}\mathcal{K}(W, R_A)$ for $\{y, z\}$, we get the known points and the solutions of a quadratic. The discriminant vanishes precisely when $\mathrm{n}\mathcal{K}(W, R, k)$ is of type $\mathrm{c}\mathcal{K}$.

Part (e) uses the same computation. The quadratic equation in (d) has constant term zero precisely when $k = 2vrt/s$. Looking at other $\mathrm{p}\mathcal{K}(W, R_B)$, $\mathrm{p}\mathcal{K}(W, R_C)$ gives the other cases listed. □

To describe a cubic of type $\mathrm{n}\mathcal{K}$ as a Grassmann cubic as above, we require two perspectors interchanged by W-isoconjugation. Provided W is not on a sideline, we need six vertices not on a sideline. After Theorem 6, there are at most six candidates, with two on each of the associated $\mathrm{p}\mathcal{K}(W, R_X)$, $X = A, B, C$. Thus, there is at most one desmic structure defining the cubic. Also from Theorem 6, there is no structure if the cubic is a $\mathrm{c}\mathcal{K}(\#F, R)$, for then the "six" points and the perspectors are all F. We need to investigate the cases where either of the final solutions in Theorem 6(b) lie on a sideline. Rather than interrupt the general argument, we postpone the discussion of these cases to an Appendix. They still contain a unique structure which can be used to generate the cubics as Grassmann loci. The structure is a degenerate kind of desmic structure. We show that, in any other case, the six points do constitute a suitable desmic structure. In the proof, we assume that we can choose an intersection of $\mathrm{n}\mathcal{K}(W, R, k)$ and $\mathrm{p}\mathcal{K}(W, R_A)$ not on a sideline, so we need the discussion of the Appendix to tidy up the remaining cases.

Theorem 7. *If a cubic $\mathcal{C}$ is of type* $\mathrm{n}\mathcal{K}$*, but not of type* $\mathrm{c}\mathcal{K}$*, then there is a unique desmic structure which defines $\mathcal{C}$ as a Grassmann cubic.*

Proof. Suppose that $\mathcal{C} = \mathrm{n}\mathcal{K}(W, R, k)$ with $W = u : v : w$, and $R = r : s : t$. We require perspectors $P_1 = p_{11} : p_{12} : p_{13}$, and $P_2 = p_{21} : p_{22} : p_{23}$ such that $r : s : t = p_{11} - p_{21} : p_{12} - p_{22} : p_{13} - p_{23}$. This amounts to two linear equations which can be used to solve for p_{22} and p_{23} in terms of p_{11}, p_{12}, p_{13} and p_{21}. We also require that $u : v : w = p_{11}p_{21} : p_{12}p_{22} : p_{13}p_{23}$. This gives three relations in p_{11} and p_{21} in terms of p_{12} and p_{13}. These are consistent provided $A' = p_{21} : p_{12} : p_{13}$ is on $\mathrm{p}\mathcal{K}(W, R_A)$. This uses a Maple calculation. We can solve for p_{11} in terms of p_{12}, p_{13} and p_{21}, provided we do not have (after scaling) $p_{12} = -u/r$, $p_{13} = v/s$ and $p_{21} = w/t$. So far, we have shown that, if A' is on $\mathrm{p}\mathcal{K}(W, R_A)$, then we can reconstruct perspectors which give rise to some $\mathrm{n}\mathcal{K}(W, R, k')$. Provided that we can choose A' also on $\mathcal{C}$, but not on a sideline, then $k' = k$ directly, so we get $\mathcal{C}$ as a Grassmann cubic. As we saw in Theorem 6, there are just two such choices of A', and these are isoconjugate, so we have just one suitable desmic structure. We could equally use a point of intersection of C with $\mathrm{p}\mathcal{K}(W, R_B)$ or with $\mathrm{p}\mathcal{K}(W, R_C)$. It follows that there is a unique desmic structure unless $\mathcal{C}$ has the points $-u/r : v/s : w/t$, $u/r : -v/s : w/t$ and $u/r : v/s : -w/t$. But then $k = 2ust/r = 2vrt/s = 2wrs/t$, so that $W = R_2$, and $k = 2rst$. It follows that C is the degenerate cubic $(ty + sz)(rz + tx)(sx + ry) = 0$. It is easy to check that this is given as a Grassmann cubic by the degenerate desmic structure with $A' = A'' = -r : s : t$, and similarly for B', B'', S' and C''. This has perspectors, desmon and harmon equal to R. The GP locus is the whole plane. □

3. Parents and children

The reader will have noted the resemblance between the equations for the cubics $GP(\Delta')$ and $GN(\Delta')$. In [2, notes on K216], Gibert observes this for $K001$ and $K216$. He refers to $K216$ as a sister of $K001$. In Theorem 7, we saw that each cubic $\mathcal{C}$ of type $\mathrm{n}\mathcal{K}$ which is not of type $\mathrm{c}\mathcal{K}$ is the Grassmann cubic associated with a unique desmic structure, and hence with a unique cubic $\mathcal{C}'$ of type $\mathrm{p}\mathcal{K}$. We call the cubic $\mathcal{C}'$ the parent of $\mathcal{C}$. On the other hand, Theorem 5 shows that a cubic $\mathcal{C}$ of type $\mathrm{p}\mathcal{K}$ contains infinitely many desmic structures, each defining a cubic of type $\mathrm{n}\mathcal{K}$. We call each of these cubics a child of $\mathcal{C}$. Our first task is to describe the children of a cubic $\mathrm{p}\mathcal{K}(W, S)$. This involves the equation of the polar conic of S, see §1. Our calculations also give information on the parents of the family of cubics of type $\mathrm{n}\mathcal{K}$ with fixed pole and root.

Theorem 8. *Suppose that $\mathcal{C} = \mathrm{p}\mathcal{K}(W, S)$ with $W = u : v : w$, and $S = r : s : t$.*
(a) *Any child of $\mathcal{C}$ is of the form $\mathrm{n}\mathcal{K}(W, R, k)$, with R on*

$$\mathrm{p}\mathcal{C}(W, S) : \qquad (vt^2 - ws^2)x^2 + (wr^2 - ut^2)y^2 + (us^2 - vr^2)z^2 = 0.$$

(b) *If R is a point of $\mathrm{p}\mathcal{C}(W, S)$ which is not S and not fixed by W-isoconjugation, then there is a unique child of $\mathcal{C}$ of the form $\mathrm{n}\mathcal{K}(W, R, k)$.*
(c) *Any cubic $\mathrm{n}\mathcal{K}(W, R, k)$ which is not of type $\mathrm{c}\mathcal{K}$ has parent of the form $\mathrm{p}\mathcal{K}(W, S)$ with S on $\mathrm{p}\mathcal{C}(W, R)$.*

(d) *If* $\mathrm{n}\mathcal{K}(W, R, k)$ *has parent* $\mathrm{p}\mathcal{K}(W, S)$*, then the perspectors are the non-trivial intersections of* $\mathrm{p}\mathcal{K}(W, R)$ *and* $\mathrm{p}\mathcal{K}(W, S)$.

Proof. (a) We know from Theorem 7 that a child $\mathrm{n}\mathcal{K}(W, R, k)$ of $\mathcal{C}$ arises from a desmic structure. Suppose the perspectors are P_1 and P_2. From Theorem 4(f) (P_1, P_2, R, S) is a harmonic range. It follows that there are constants m and n with $P_1 = mR + nS$ and $P_2 = -mR + nS$. From Theorem 4(a), W is the barycentric product of P_1 and P_2. Suppose that $R = x : y : z$. Then we have

$$\frac{m^2x^2 - n^2r^2}{u} = \frac{m^2y^2 - n^2s^2}{v} = \frac{m^2z^2 - n^2t^2}{w}.$$

If we eliminate m^2 and n^2 from these, we get $\mathrm{p}\mathcal{C}(W, S) = 0$.

(b) Given such an R, we can reverse the process in (a) to obtain a suitable value for $(m/n)^2$. Choosing either root, we get the required perspectors.

(c) is really just the observation that S is on $\mathrm{p}\mathcal{C}(W, R)$ if and only if R is on $\mathrm{p}\mathcal{C}(W, S)$.

(d) In Theorem 4, we noted that the perspectors lie on $\mathrm{p}\mathcal{K}(W, S)$ and on $\mathrm{p}\mathcal{K}(W, R)$. Now these cubics meet at A, B, C and the four points fixed by W-isoconjugation. There must be just two other (non-trivial) intersections. □

Example 1. In terms of triangle centers, the most prolific parent seems to be the Neuberg cubic = $\mathrm{p}\mathcal{K}(K, X(30))$, Gibert's $K001$.

The polar cubic $\mathrm{p}\mathcal{C}(K, X(30))$ is mentioned in [5] in the discussion of its center, the Tixier point, X476. There, it is noted that it is a rectangular hyperbola passing through I, the excenters, and $X(30)$. Of course, being rectangular, the other infinite point must be $X(523)$. Using the information in [5], we see that its asymptotes pass through $X(74)$ and $X(110)$. The perspectors P_1, P_2 of desmic structures on $K001$ must be its isogonal pairs other than $\{X(30),\ X(74)\}$.

By Theorem 4(f), the root of the child cubic must be the mid-point of P_1 and P_2. The pair $\{O,\ H\}$ gives a cubic of the form $\mathrm{n}\mathcal{K}(K, X(5), k)$. The information in [2] identifies it as $K216$. The pair $\{X(13),\ X(15)\}$ gives a cubic of the form $\mathrm{n}\mathcal{K}(K, X(396), k)$. The pair $\{X(14),\ X(16)\}$ gives a cubic of the form $\mathrm{n}\mathcal{K}(K, X(395), k)$.

As noted above, $X(523)$ is on the polar conic, so we also have a child of the form $\mathrm{n}\mathcal{K}(K, X(523), k)$. Since $X(523)$ is not on the cubic, the perspectors must be at infinity. As they are isogonal conjugates, they must be the infinite circular points. These have already been noted as lying on $K001$. We now have additional centers on $\mathrm{p}\mathcal{C}(K, X(30))$, $X(5)$, $X(395)$, $X(396)$, as well as $X(1)$, $X(30)$, $X(523)$. We also have the harmonic associates of each of these points!

4. Roots and pivots

If we have a cubic $\mathrm{n}\mathcal{K}(W, R, k)$ defined by a desmic structure, then it has a parent cubic $\mathrm{p}\mathcal{K}(W, S)$. From Theorem 8(c), we know that S is on $\mathrm{p}\mathcal{C}(W, R)$. Since R is also on the conic, we can identify S from an equation for RS. Although the results were found by heavy computations, we can establish them quite simply by "guessing" the pole of RS with respect to $\mathrm{p}\mathcal{C}(W, R)$.

Theorem 9. *Suppose that* $W = u : v : w$, $R = r : s : t$ *and* k *are such that* $\mathrm{n}\mathcal{K}(W, R, k)$ *is defined by a desmic structure. Let* $\mathrm{p}\mathcal{K}(W, S)$ *be the parent of* $\mathrm{n}\mathcal{K}(W, R, k)$.
(a) *The line* RS *is the polar of* $P = 2ust - kr : 2vtr - ks : 2wrs - kt$ *with respect to* $\mathrm{p}\mathcal{C}(W, R)$.
(b) *The point* S *has first barycentric coordinate* $4r(-r2vw + s2wu + t2uv) - 4kstu + k2r$.

Proof. Suppose that the desmic structure has normalized perspectors $P_1 = f : g : h$ and $P_2 = f' : g' : h'$. Then we have

$$\begin{array}{lll} r = f - f', & s = g - g', & t = h - h'; \\ u = ff', & v = gg', & w = hh', \\ k = 2(f'g'h' - fgh). & & \end{array}$$

(a) The first barycentric of P is then

$$2(ff'(g - g')(h - h') - (f'g'h' - fgh)(f - f')) = 2(fg - f'g')(fh - f'h').$$

The coefficient of x^2 in $\mathrm{p}\mathcal{C}(W, R)$ is

$$vt^2 - ws^2 = (gh - g'h')(hg' - gh').$$

If we discard a symmetric factor the polar of P is the line P_1P_2, *i.e.*, the line RS.

(b) We know that $S = f + f' : g + g' : h + h'$. If we substitute the above values for r, s, t, u, v, w, k, the expression becomes $K(f + f')$, where K is symmetric in the variables. Thus, S is as stated. □

This result allows us to find the parent of a cubic, even if we cannot find the perspectors explicitly. It would also allow us to find the perspectors since these are the W-isoconjugate points on the line RS. Thus the perspectors arise as the intersections of a line and (circum)conic.

Example 2. The second Brocard cubic $\mathrm{n}\mathcal{K}_0(K, X(523))$, Gibert's $K018$. We cannot identify the perspectors of the desmic structure. They are complex. Theorem 9(b) gives the parent as $\mathrm{p}\mathcal{K}(K, X(5))$ - the Napolean cubic, and Gibert's $K005$.

Example 3. The kjp cubic $\mathrm{n}\mathcal{K}_0(K, K)$, Gibert's $K024$.

Theorem 9(b) gives the parent as $\mathrm{p}\mathcal{K}(K, O)$ - the McCay cubic and Gibert's $K003$. Theorem 9(a) gives RS as the Brocard axis. It follows that the perspectors of the desmic structure are the intersections of the Brocard axis with the Kiepert hyperbola.

Our next result identifies the children of a given $\mathrm{p}\mathcal{K}(W, R)$ which are of the form $\mathrm{n}\mathcal{K}_0(W, R)$. It turns out that the perspectors must lie on another cubic $\mathrm{n}\mathcal{K}_0(W, T)$. The root T is most neatly defined using the generalization of Gibert's PK-transform.

Theorem 10. *Suppose that* $\{P, P^*\} \neq \{S, S^*\}$ *are a pair of* W*-isoconjugates on* $\mathrm{p}\mathcal{K}(W, S)$. *Then they define a desmic structure with associated cubic of the form* $\mathrm{n}\mathcal{K}_0(W, R)$ *if and only if* P, P^* *lie on* $\mathrm{n}\mathcal{K}_0(W, PK(W, S))$.

Proof. Suppose that $P = x : y : z$, $S = r : s : t$, $W = u : v : w$. Then the point P^* is $u/x : v/y : w/z$. As P is on $\mathrm{p}\mathcal{K}(W, S)$, there exist constants m, n with $P + mP^* = nS$. Then the normalized forms for the perspectors of the desmic structure are P and mP^*, and $R = P - mP^*$. If we look at a pair of coordinates in the expression $P + mP^* = nS$, we get an expression for m in terms of two of x, y, z. These are

$$m_x = \frac{(yt - zt)yx}{ysw - ztv}, \quad m_y = \frac{(zr - xt)xz}{ztu - xrv}, \quad m_z = \frac{(xs - yr)xy}{xrv - ysu}.$$

We can now compute the k such that the cubic is $\mathrm{n}\mathcal{K}(W, R, k)$ as

$$2\left(\frac{m_x m_y m_z uvw}{xyz} - xyz\right).$$

We have an $\mathrm{n}\mathcal{K}_0$ if and only if this vanishes. Maple shows that this happens precisely when P (and hence P^*) is on the cubic $\mathrm{n}\mathcal{K}_0(W, T)$, where

$$T = ur(vt^2 - ws^2) : vs(wr^2 - ut^2) : wt(us^2 - vr^2) = PK(W, S).$$

$\square$

Note that we get an $\mathrm{n}\mathcal{K}_0$ when P, P^* lie on $\mathrm{p}\mathcal{K}(W, S)$ and $\mathrm{n}\mathcal{K}_0(W, PK(W, S))$. Then there are three pairs and three cubics. Also, $PK(W, S) = PK(W, S^*)$ - see §5 - so $\mathrm{p}\mathcal{K}(W, S)$ and $\mathrm{p}\mathcal{K}(W, S^*)$ give rise to the same $\mathrm{n}\mathcal{K}_0$.

Example 4. Applying Theorem 10 to the McCay cubic $\mathrm{p}\mathcal{K}(K, O)$ and the Orthocubic $\mathrm{p}\mathcal{K}(K, H)$ we get the second Brocard cubic $K019 = \mathrm{n}\mathcal{K}_0(K, X(647))$. In the former case, we can identify the perspectors of the desmic structures. From [2, K019], we know that the points of $K019$ are the foci of inconics with centers on the Brocard axis OK. Also, if P is on the McCay cubic, then PP^* passes through O. If PP^* is not OK, then the center is on PP^* and on OK, so must be O. This gives four of the intersections, two of which are real. Otherwise, P and P^* are the unique K-isoconjugates on OK. These are the intersections of the Brocard axis and the Kiepert hyperbola. Again these are complex. They give the cubic $\mathrm{n}\mathcal{K}_0(K, K)$ - see Example 3. In §5, we meet these last two points again.

Example 5. The Thomson cubic $\mathrm{p}\mathcal{K}(K, G)$ and the Grebe cubic $\mathrm{p}\mathcal{K}(K, K)$ give the cubic $\mathrm{n}\mathcal{K}_0(K, X(512))$. We will meet this cubic again in §5.

5. Desmic structures with triply perspective triangles

As we saw in Theorem 3, a cubic of type $\mathrm{n}\mathcal{K}_0$ arises from a desmic structure in which the triangles are triply perspective with the reference triangle $\triangle ABC$. We begin with a discussion of an obvious way of constructing such a structure. It turns out that almost all triply perspective structures arise in this way.

Lemma 11. *Suppose that $P = p : q : r$ and $Q = u : v : w$ are points with distinct cevians.*
(a) *Let*

$$\begin{aligned} A' &= BP \cap CQ, & B' &= CP \cap AQ, & C' &= AP \cap BQ, \\ A'' &= BQ \cap CP, & B'' &= CQ \cap AP, & C'' &= AQ \cap BP. \end{aligned}$$

Then the desmic structure with vertices A, B, C, A', B', C', A'', B'', C'', and perspectors $P_1 = \frac{1}{qw} : \frac{1}{ru} : \frac{1}{pv}$ and $P_2 = \frac{1}{rv} : \frac{1}{pw} : \frac{1}{qu}$ has triangles $\triangle ABC$, $\triangle A'B'C'$, $\triangle A''B''C''$ which are triply perspective.
(b) *Each desmic structure including the vertices A, B, C in which*
(i) *the triangles are triply perspective, and*
(ii) *the perspectors are distinct arises from a unique pair P, Q with $P \neq Q$, and perspectors normalized as in (a).*

Proof. (a) We begin by looking at the desmic structure which is derived from the given P_1 and P_2 as in Theorem 4. Thus, the first three vertices are obtained by replacing a coordinate of P_1 by the corresponding coordinate of P_2. This has the vertices named, for example the first vertex is $\frac{1}{rv} : \frac{1}{ru} : \frac{1}{pv}$. This is on BP and CQ, so is A'. It is easy to see that the triangles doubly perspective, with perspectors P and Q, and hence triply perspective.

(b) Suppose we are given such a desmic structure. Then the perspectors are $P_1 = f : g : h$ and $P_2 = f' : g' : h'$ with$fgh = f'g'h'$ (see Lemmata 2 and 3). We find points P and Q which give rise to these as in part (a). From the coordinates of P_1, we can solve for u, v and w in terms of p, q, r and the coordinates of P_1. Then, using two of the coordinates of P_2, we can find q and r in terms of p. The equality of the third coordinates follows from the condition $fgh = f'g'h'$. As the perspectors are distinct, $P \neq Q$. □

Note that from the normalized perpsectors, we can recover the vertices, even if some cevians coincide. For example, A' is $\frac{1}{rv} : \frac{1}{ru} : \frac{1}{pv}$.

Definition 4. *The desmic structure defined in Lemma 11 is denoted by $\mathcal{D}(P, Q)$.*

Theorem 12. *If P and Q are triangle centers with functions $p(a, b, c)$ and $q(a, b, c)$, then the desmic structure $\mathcal{D}(P, Q)$ has*
(a) *perspectors P_1, P_2 with functions $h(a, b, c) = \frac{1}{p(b,c,a)q(c,a,b)}$ and $h(a, c, b)$.*
(b) *$\{P_1,\ P_2\}$ is a bicentric pair.*
(c) *The vertices of the triangles are $[h(a, c, b), h(b, c, a), h(c, a, b)]$, and so on.*

The proof requires only the observation that, as P and Q are centers, $p(a, b, c) = p(a, c, b)$ and $q(a, b, c) = q(a, c, b)$.

We leave it as an exercise to the reader that, if $\{P, Q\}$ is a bicentric pair, then P_1, P_2 are centers. It follows that, if P, Q are centers and $\mathcal{D}(P, Q)$ has perspectors P_1, P_2, then $\mathcal{D}(P_1, P_2)$ has triangle centers P', Q' as perspectors. As further exercises, the reader may verify that P', Q' are the Q^2-isoconjugate of P and the P^2-isoconjugate of Q. The desmon of the second structure is the P-Hirst inverse of Q.

We can compute the equations of the cubics from the information in Lemma 11(a). These involve ideas introduced in our Definition 2.

Theorem 13. *Suppose that $P \neq Q$. The cubics associated with the desmic structure $\mathcal{D}(P, Q)$ are $\mathrm{p}\mathcal{K}(W, NK(W, P))$ and $\mathrm{n}\mathcal{K}_0(W, PK(W, P))$, where W is the isoconjugation which interchanges P and Q.*

Proof. We have the perspectors P_1 and P_2 of the desmic structure from Lemma 11(a). These give isoconjugation as that which interchanges P_1 and P_2 - see Theorem 4. Now observe that this also interchanges P and Q, so $Q = P^*$. From Theorem 4, we have coordinates for the desmon S and the harmon R in terms of those of P_1 and P_2. Using our formulae for P_1 and P_2, we get the stated values of S and R. □

Definition 5. *Suppose that $P \neq Q$, and that $\mathcal{C} = \mathrm{n}\mathcal{K}_0(W, R)$ is associated with the desmic structure $\mathcal{D}(P, Q)$. Then*
(a) *P, Q are the cevian points for $\mathcal{C}$.*
(b) *The perspectors of $\mathcal{D}(P, Q)$ are the Grassmann points for $\mathcal{C}$.*

Theorem 14. *Suppose that $\mathcal{C} = \mathrm{n}\mathcal{K}_0(W, R)$ is not of type $\mathrm{c}\mathcal{K}$, and does not have $W = R^2$.*
(a) *The cevian points for $\mathcal{C}$ are the W-isoconjugate points on $\mathcal{T}(R^*)$. These are the intersections of $\mathcal{C}(R)$ and $\mathcal{T}(R^*)$.*
(b) *The Grassmann points for $\mathcal{C}$ are the W-isoconjugate points on the polar of $PK(W, R)$ with respect to $\mathcal{C}(R)$.*

Proof. From Theorems 3 and 7, we know that $\mathcal{C}$ is a Grassmann cubic associated with a desmic structure which has triply perspective triangles. If the perspectors of the structure coincide at X, then the equation shows that $R = X$ and $W = X^2$. But we assumed that $W \neq R^2$, so we do not have this case.

(a) From Theorem 13, this structure is $\mathcal{D}(P, Q)$ with $Q = P^*$, and $R = PK(W, P) = PK(W, Q)$. From a remark following Definition 2, P and Q lie on the conic $\mathcal{C}(R)$. Since they are W-isoconjugates, they also lie on the W-isoconjugate of $\mathcal{C}(R)$. This is $\mathcal{T}(R^*)$. The conic and line have just two intersections, so this gives precisely the pair $\{P, Q\}$. These are precisely the pair of W-isoconjugates on $T(R^*)$.

(b) By Theorem 4, the Grassmann points are W-isoconjugate and lie on RS, where S is the desmon of the desmic structure. As $\mathcal{C}$ is an $\mathrm{n}\mathcal{K}_0$, Theorem 9 gives S as a point which we recognize as the pole of $\mathcal{T}(R^*)$ with respect to the conic $\mathcal{C}(R)$. Now R is the pole of $\mathcal{T}(R)$ for this conic, so RS is the polar of $\mathcal{T}(R) \cap \mathcal{T}(R^*) = PK(W, R)$. □

In Theorem 14, we ignored cubics of type $\mathrm{n}\mathcal{K}(R^2, R, k)$. To make the algebra easier, we replace the constant k by $k'rst$, where $R = r : s : t$.

Theorem 15. *If $\mathcal{C} = \mathrm{n}\mathcal{K}(R^2, R, k'rst)$ is not of type $\mathrm{c}\mathcal{K}$, then $\mathcal{C}$ is the Grassmann cubic associated with the desmic structure having perspectors R and aR, where a is a root of*

$$x^2 + \left(1 + \frac{k'}{2}\right)x + 1 = 0.$$

When $k' = 2$, $a = -1$, the desmic structure and $\mathcal{C}$ are degenerate.

When $k' \neq 2, -6$, the corresponding $\mathrm{p}\mathcal{K}$ is $\mathrm{p}\mathcal{K}(R^2, R)$, which is the union of the R-cevians.

Proof. When we use Maple to solve the equations to identify A' and A'', we discover them as $r : as : at$, with a as above. This identifies the perspectors as R and aR. It is easy to verify that this choice leads to $\mathcal{C}$. Note that the two roots are inverse, so we get the same vertices from either choice. The quadratic has equal roots when $k' = 2$ or -6. The latter gives the c$\mathcal{K}$ in the class. In this case, the "cubic" equation is identically zero as $a = 1$. In the former case, $a = -1$, so we do get the n$\mathcal{K}$ as a Grassmann cubic, but the equation factorizes as $(ty + sz)(rz + tx)(sx + ry) = 0$. In the "desmic structure" $A' = A''$, $B' = B''$, $C' = C''$ as $a = -1$.

Note that here the cubic is not of type c$\mathcal{K}$ as it contains three fixed points R_A, R_B, R_C of R^2-isoconjugation. When $k' \neq 2,\ -6$, the desmon is defined, and is again R. Finally, p$\mathcal{K}(R^2, R)$ is $(ty - sz)(rz - tx)(sx - ry) = 0$. □

Example 6. The third Brocard cubic, n$\mathcal{K}_0(K, X(647))$, is $K019$ in Gibert's list. As it is of type n$\mathcal{K}_0$, but not of type c$\mathcal{K}$, Theorem 12(b) applies. The conic $\mathcal{C}(R)$ is the Jerabek hyperbola. Also, R^* is $X(648)$, so that the line $\mathcal{T}(R^*)$ is the Euler line. These intersect in (the K-isoconjugates) O and H. This gives some new points on the cubic:

$$\begin{array}{lll} A' = BO \cap CH, & B' = CO \cap AH, & C' = AO \cap BH, \\ A'' = BH \cap CO, & B'' = CH \cap AO, & C'' = AH \cap BO. \end{array}$$

Also, the cubic can be described as the Grassmann cubic $GN(\triangle A'B'C')$ or $GN(\triangle A''B''C'')$. The associated GP has the pivot $NK(K, O) = NK(K, H) = X(185)$. The cubic p$\mathcal{K}(K, X(185))$ is not (yet) listed in [2], but we know that it has the points A', B', C', A'', B'', C'', the perspectors P_1 and P_2, as well as I, the excenters and $X(185)$.

Example 7. The first Brocard cubic, n$\mathcal{K}_0(K, X(385))$, is $K017$ in Gibert's list. Here, the vertices A', B', C', A'', B'', C'' have already been identified. They are the vertices of the first and third Brocard triangles. The points P and Q are the Brocard points. The associated GP is p$\mathcal{K}(K, X(384))$, the fourth Brocard cubic and $K020$ in Gibert's list. Again, Gibert's website shows the points on $K020$, together with $\{P_1,\ P_2\} = \{X(32),\ X(76)\}$.

Example 8. The cubics p$\mathcal{K}(K, X(39))$ and n$\mathcal{K}_0(K, X(512))$. These cubics do not appear in Gibert's list, but the associated desmic structure is well-known. Take $= G$, $Q = K$ in Lemma 11(a). Again we get isogonal cubics. The vertices of the structure are the intersections of medians and symmedians. From the proof of Lemma 11, P_1, P_2 are the Brocard points. This configuration is discussed in, for example, [4], but the proof that the triangles obtained from the intersections are perspective with $\triangle ABC$ uses special properties of G and K. Here, our Lemma 8(a) gives a very simple (geometric) proof of the general case. Here, $PK(K, G) = PK(K, K) = X(512)$, and $NK(K, G) = NK(K, K) = X(39)$.

The points P and Q for a cubic of type n$\mathcal{K}_0$ are identified as the intersections of a line with a conic. Of course, it is possible that these have complex coordinates. This happens, for example, for the second Brocard cubic, n$\mathcal{K}_0(K, X(523))$. There, the line is the Brocard axis and the conic is the Kiepert hyperbola. A Cabri sketch shows that these do not have real intersections. We have met these intersections in §4.

Example 9. The kjp cubic $\mathrm{n}\mathcal{K}_0(K, K)$ is $K024$ in Gibert's list. The desmic structure is $\mathcal{D}(P, Q)$, where P, Q are the intersections of the line at infinity and the circumcircle. These are the infinite circular points. We met this cubic in Example 3, where we identified the perspectors of the desmic structure as the intersections of the Brocard axis with the Kiepert hyperbola.

6. Harmonic associates and other cubics

In the Introduction, we introduced the idea of harmonic associates. This gives a pairing of our cubics. We begin with a result which relates desmic structures. This amplifies remarks made in the proof of Theorem 6.

Theorem 16. *Suppose that D is a desmic structure with normalized perspectors P_1, P_2, and cubics $\mathrm{n}\mathcal{K}(W, R, k)$, $\mathrm{p}\mathcal{K}(W, S)$. Then the desmic structure D' with normalized perspectors P_1, $-P_2$ has*
(a) *vertices the harmonic associates of those of D, and*
(b) *cubics $\mathrm{n}\mathcal{K}(W, S, k')$, $\mathrm{p}\mathcal{K}(W, R)$.*

Proof. We refer the reader to the proof of Theorem 6(a), which in turn uses the notation of Theorem 4(a). □

We refer to the cubic $\mathrm{n}\mathcal{K}(W, S, k')$ obtained in this way as the harmonic associate of $\mathrm{n}\mathcal{K}(W, R, k)$.

Corollary 17. *If $\mathrm{n}\mathcal{K}(W, R, k) = GN(\Delta)$, then $\mathrm{p}\mathcal{K}(W, R) = GP(\Delta')$, where Δ' is the harmonic associate of Δ.*

Example 10. The second Brocard cubic $\mathrm{n}\mathcal{K}(K, X(385)) = K017$ was discussed in Example 7. It is $GN(\Delta)$, where Δ is either the first or third Brocard triangle. From Corollary 17, the cubic $\mathrm{p}\mathcal{K}(K, X(385)) = K128$ is $GP(\Delta')$, where Δ' is the harmonic associate of either of these triangles. This gives us six new points on $K128$. The cubic $\mathrm{n}\mathcal{K}_0(K, X(512))$ was introduced in Example 8. It is $GN(\Delta)$, where Δ is formed from intersections of medians and symmedians. From the Corollary, the fifth Brocard cubic $\mathrm{p}\mathcal{K}(K, X(512)) = K021$ is $GP(\Delta')$. Now the Grassmann points for $\mathrm{n}\mathcal{K}_0(K, X(512))$ are the Brocard points, so these lie on $K021$, as do the vertices of Δ'.

Example 11. Let $\mathcal{C} = K216$ of [2]. This was mentioned in §3. It is of the form $\mathrm{n}\mathcal{K}(K, X(5), k)$ with parent $\mathrm{p}\mathcal{K}(K, X(30)) = K001$. From Theorem 16, the harmonic associate is of the form $\mathrm{n}\mathcal{K}(K, X(30), k')$ with parent $\mathrm{p}\mathcal{K}(K, X(5)) = K005$. Using Theorem 9(b), a calculation shows that $K067 = \mathrm{n}\mathcal{K}(K, X(30), k'')$ has parent $K005$. From Theorem 8(b), $K005$ has a unique child with root $X(30)$. Thus $K067$ is the harmonic associate of $K216$. This gives us six points on $K067$ as harmonic associates of the points identified in [2] as being on $K216$. We will give a geometrical description of these shortly.

Three of the vertices of the desmic structure for $K216$ are the reflections of the vertices A, B, C in BC, CA, AB. These give a triangle with perspector H. We can generalize these to a general X as the result of extending each X-cevian by its own length. If $X = x : y : z$, then, starting from A, we get the point $y+z : -2y : -2z$. The A-harmonic associate is $y+z : 2y : 2z$, the intersection of the H-cevian at A with the parallel to BC through G. This suggests the following definition.

Definition 6. *For $X = x : y : z$, the desmic structure $\mathcal{D}(X)$ is that with normalized perspectors $y+z : z+x : x+y$ and $-2x : -2y : -2z$.*

Note that the first perspector is the complement $\mathsf{c}X$ of X and the second is X. We can summarize our results on such structures as follows.

Theorem 18. *Suppose that $X = x : y : z$. The cubics associated with $\mathcal{D}(X)$ are $\mathrm{n}\mathcal{K}(W, R, k)$ and $\mathrm{p}\mathcal{K}(W, S)$, where*
(i) *$W = x(y+z) : y(z+x) : z(x+y)$, the center of the inconic with perspector X,*
(ii) *$R = 2x+y+z : x+2y+z : x+y+2z$, the mid-point of X and $\mathsf{c}X$,*
(iii) *$S = -2x+y+z : x-2y+z : x+y-2z$, the infinite point on GX.*
The harmonic associate of $\mathrm{n}\mathcal{K}(W, R, k)$ passes through G, the infinite point of $\mathcal{T}(X)$, and their W-isoconjugates.

Proof. The coordinates of W, R and S follow at once from those of the perspectors and Theorem 4. The final part needs an equation for the harmonic associate. This is given by Theorem 4. The fact that G and $x(y-z) : y(z-x) : z(x-y)$ lie on the cubic is a simple verification using Maple. □

For $X = H$, we get $K216$ and $K001$ and their harmonic associates $K067$ and $K005$. The desmic structures and the points given in Theorem 18 account for most of the known points on $K216$ and $K067$.

There is one further example in [2]. In the notes on $K022 = \mathrm{n}\mathcal{K}(O, X(524), k)$, it is observed that the cubic contains the vertices of the second Brocard triangle, and hence their O-isoconjugates. The latter are the the intersections of the $X(69)$-cevians with lines through G parallel to the corresponding sidelines. These are the harmonic associates of three vertices of $\mathcal{D}(X(69))$. The other perspector is $K = \mathsf{c}X(69)$. The mid-point of $X(69)$ and K is $X(141)$, so $K022$ is the harmonic associate of the cubic $\mathrm{n}\mathcal{K}(O, X(141), k')$ with parent $\mathrm{p}\mathcal{K}(O, X(524))$. These cubics contain the vertices of $\mathcal{D}(X(69))$, including the harmonic associates of the second Brocard triangle. Also, the parent of $K022$ is $\mathrm{p}\mathcal{K}(O, X(141))$.

In the Introduction, we mentioned that the Darboux cubic $\mathrm{p}\mathcal{K}(K, X(20))$ is $GP(\Delta)$, where Δ has vertices the infinite points on the altitudes. Of course, as Δ is degenerate, $GN(\Delta)$ degenerates. It is the union of the circumcircle and the line at infinity. The harmonic associate Δ' has vertices the mid-points of the altitudes, and this leads to an $\mathrm{n}\mathcal{K}(K, X(20), k)$, and its parent which is the Thomson cubic $\mathrm{p}\mathcal{K}(K, G) = K002$. This will follow from our next result. The fact that the mid-points lie on $K002$ is noted in [5], but now we know that those points can be used to generate $K002$ as a locus of type GP. We can replace the vertices of Δ or

Δ by their isogonal conjugates. In the case of Δ the isogonal points lie on the circumcircle. For any point $X = x : y : z$, the mid point of the cevian at A is $y + z : y : z$. We make the following definition.

Definition 7. *For $X = x : y : z$, the desmic structure $\mathcal{E}(X)$ is that with normalized perspectors $y + z : z + x : x + y$ and $x : y : z$.*

Theorem 19. *Suppose that $X = x : y : z$. Let Δ be the triangle $\triangle A'B'C'$ of $\mathcal{E}(X)$. Then $GN(\Delta) = \mathrm{n}\mathcal{K}(W, R, k)$ and $GP(\Delta) = \mathrm{p}\mathcal{K}(W, G)$, where*
$W = x(y + z) : y(z + x) : z(x + y)$, the complement of the isotomic conjugate of X,
$R = -x + y + z : x - y + z : x + y - z$, the anticomplement of X,
$k = 2((y + z)(z + x)(x + y) - xyz)$.

The harmonic associates are $GN(\Delta') = \mathrm{n}\mathcal{K}(W, G, k')$, which degenerates as $\mathcal{C}(W)$ and the line at infinity, and $GP(\Delta') = \mathrm{p}\mathcal{K}(W, R)$, which is a central cubic with center the complement of X.

Most of the result follow from the equations given by Theorem 4. The fact that $\mathrm{p}\mathcal{K}(W, R)$ is central is quite easy to check, but it is a known result. In [6], Yiu shows that the cubic defined by $GP(\Delta')$ has the given center. Yiu derives interesting geometry related to $\mathrm{p}\mathcal{K}(W, R)$, and these are summarized in [1, §3.1.3]. The case $X = H$ gives $W = K$, $R = X(20)$, so we get $GP(\Delta) = K002$, the Thomson cubic, and $GP(\Delta') = K004$, the Darboux cubic.

Remarks. (1) From [1, Theorem 3.1.2], we know that if $W \neq G$, there is a unique central $\mathrm{p}\mathcal{K}$ with pole W. After Theorem 19, this arises from the desmic structure $\mathcal{E}(X)$, where X is the isotomic conjugate of the anticomplement of W. The center is then the complement of X, and hence the G-Ceva conjugate of W. It is also the perspector of $\mathcal{E}(X)$ other than X. The pivot of the central $\mathrm{p}\mathcal{K}$ is then the anticomplement of X, and hence the anticomplement of the isotomic conjugate of the anticomplement of the pole.

(2) The cubic $\mathrm{p}\mathcal{K}(W, G)$ clearly contains G and W. From the previous remark, it contains the G-Ceva conjugate of W and its W-isoconjugate (the point X). It also includes the mid-points of the X-cevians and their W-isoconjugates. The last six are the vertices of a defining desmic structure. Finally, it includes the mid-points of the sides of $\triangle ABC$.

We have seen that there are several pairs of cubics of type $\mathrm{p}\mathcal{K}$ which are loci of type GP from harmonic associate triangles. We can describe when this is possible.

Theorem 20. *For a given W, suppose that R and S are distinct points, neither fixed by W-isoconjugation.*
(a) *There exist harmonic triangles Δ and Δ' with $\mathrm{p}\mathcal{K}(W, R) = GP(\Delta)$ and $\mathrm{p}\mathcal{K}(W, S) = GP(\Delta')$ if and only if $\mathrm{p}\mathcal{C}(W, R) = \mathrm{p}\mathcal{C}(W, S)$.*
(b) *If $\mathrm{p}\mathcal{C}(W, R) = \mathrm{p}\mathcal{C}(W, S)$, then the Grassmann points are*
(i) *the non-trivial intersections of $\mathrm{p}\mathcal{K}(W, R)$ and $\mathrm{p}\mathcal{K}(W, S)$,*
(ii) *the W-isoconjugate points on RS.*

Proof. (a) If $\mathrm{p}\mathcal{K}(W,R)$ and $\mathrm{p}\mathcal{K}(W,S)$ are loci of the given type, then $GN(\Delta') = \mathrm{n}\mathcal{K}(W,R,k)$ by Theorem 16. Thus this $\mathrm{n}\mathcal{K}$ has parent $\mathrm{p}\mathcal{K}(W,S)$. By Theorem 8, $\mathrm{p}\mathcal{C}(W,R) = \mathrm{p}\mathcal{C}(W,S)$. Now suppose that $\mathrm{p}\mathcal{C}(W,R) = \mathrm{p}\mathcal{C}(W,S)$. Then R is on $\mathrm{p}\mathcal{C}(W,S)$. By Theorem 8, there is a unique child of $\mathrm{p}\mathcal{K}(W,R)$ of the form $\mathrm{n}\mathcal{K}(W,S,k')$. From Theorem 7, $\mathrm{n}\mathcal{K}(W,S,k') = GN(\Delta)$ for a triangle Δ, and $GP(\Delta) = \mathrm{p}\mathcal{K}(W,R)$. By Theorem 16, $GP(\Delta') = \mathrm{p}\mathcal{K}(W,S)$.

(b) The Grassmann points are the same for Δ and Δ', and lie on both cubics. There are just two non-trivial points of intersection, so these are the Grassmann points. The Grassmann points are W-isoconjugate, and lie on RS, giving (ii). □

We already have some examples of pairs of this kind:
$K001$ and $K005$ with Grassmann points O, H. The desmic structures are those for $K067$ and $K216$.
$K002$ and $K004$ with Grassmann points O, H. The desmic structures appear above.
$K020$ and $K128$ with Grassmann points $X(32)$, $X(76)$. See the first of Example 10.

From Example 1, we found children of $K001$ with roots $X(395)$, $X(396)$, $X(523)$. We then have
$K001$ and $K129a = \mathrm{p}\mathcal{K}(K, X(395))$ with Grassmann points $X(14)$, $X(16)$;
$K001$ and $K129b = \mathrm{p}\mathcal{K}(K, X(396))$ with Grassmann points $X(13)$, $X(15)$;
$K001$ and $\mathrm{p}\mathcal{K}(K, X(523))$ with Grassmann points the infinite circular points.

In Example 3, we showed that $K024 = \mathrm{n}\mathcal{K}_0(K,K)$ is a child of $K003$. We therefore have $K003$ and $K102 = \mathrm{p}\mathcal{K}(K,K)$, the Grebe cubic. The Grassmann points are the intersection of the Brocard axis and the Kiepert hyperbola. These are complex.

7. Further examples

So far, almost all of our examples have been isogonal cubics. In this section, we look at some cubics with different poles. We have chosen examples where at least one of the cubics involved is in [2]. In the latest edition of [2], we have the class $CL041$. This includes cubics derived from $W = p : q : r$. In our notation, we have the cubic $\mathrm{n}\mathcal{K}_0(W,R)$, where $R = p^2 - qr : q^2 - pr : r^2 - pq$, the G-Hirst inverse of W. The parent is $\mathrm{p}\mathcal{K}(W,S)$, where $S = p^2 + qr : q^2 + pr : r^2 + pq$. The Grassmann points are the barycentric square and isotomic conjugate of W. The cevian points are the bicentric pair $1/r : 1/p : 1/q$ and $1/q : 1/r : 1/p$. Example 7 is the case where $W = K$, so that the Grassmann points are the centers $X(32)$, $X(76)$, and the cevian points are the Brocard points. Our first two examples come from $CL041$.

Example 12. If we put $W = X(1)$ in construction $CL041$, we get $\mathrm{n}\mathcal{K}_0(X(1), X(239))$ with parent $K132 = \mathrm{p}\mathcal{K}(X(1), X(894))$. The Grassmann points are K, $X(75)$, and the cevian points are those described in [3] as the Jerabek points. A harmonic associate of $K132$ is then $K323 = \mathrm{p}\mathcal{K}(X(1), X(239))$.

Example 13. If we put $W = H$ in construction of $CL041$, we get $\mathrm{n}\mathcal{K}_0(H, X(297))$ with parent $\mathrm{p}\mathcal{K}(H, S)$, where S is as given in the discussion of $CL041$. The Grassmann points are $X(393)$, $X(69)$, and the cevian points are those described in [3] as the cosine orthocenters.

Example 14. The shoemaker's cubics are $K070a = \mathrm{p}\mathcal{K}(H, X(1586))$ and $K070b$ = $\mathrm{p}\mathcal{K}(H, X(1585))$. As stated in [2], these meet in G and H. If we normalize G as $1 : 1 : 1$ and H as $\lambda \tan A : \lambda \tan B : \lambda \tan C$, with $\lambda = \pm 1$, we get an $\mathrm{n}\mathcal{K}(H, X(1585), k)$ with parent $K070a$, and an $\mathrm{n}\mathcal{K}(H, X(1585), k')$ with parent $K070b$. The Grassmann points are G and H. Note that $K070a$ and $K070b$ are therefore harmonic associates.

The next five examples arise from Theorem 19. Further examples may be obtained from the page on central $\mathrm{p}\mathcal{K}$ in [2].

Example 15. The complement of $X(1)$ is $X(10)$, the Spieker center, the anticomplement of $X(1)$ is $X(8)$, the Nagel point. The center of the inconic with perspector $X(1)$ is $X(37)$. If we apply Theorem 19 with $X = X(1)$, then we get a cubic $\mathcal{C} = \mathrm{n}\mathcal{K}(X(37), X(8), k)$ with parent $\mathrm{p}\mathcal{K}(X(37), G)$. The Grassmann points are $X(1)$ and $X(10)$. The cubic $\mathrm{p}\mathcal{K}(X(37), G)$ does not appear in the current [2], but contains $X(1)$, $X(2)$, $X(10)$ and $X(37)$. The harmonic associate of C degenerates as the line at infinity and the circumconic with perspector $X(37)$, and has parent $K033 = \mathrm{p}\mathcal{K}(X(37), X(8))$.

Example 16. The complement of $X(7)$, the Gergonne point, is $X(9)$, the Mittenpunkt, the anticomplement of $X(7)$ is $X(144)$. The center of the inconic with perspector $X(7)$ is $X(1)$. If we apply Theorem 19 with $X = X(7)$, then we get a cubic $\mathcal{C} = \mathrm{n}\mathcal{K}(X(1), X(144), k)$ with parent $\mathrm{p}\mathcal{K}(X(1), G)$. The Grassmann points are $X(7)$ and $X(9)$. The cubic $\mathrm{p}\mathcal{K}(X(1), G)$ does not appear in the current [2], but contains $X(1)$, $X(2)$, $X(7)$ and $X(9)$. The harmonic associate of $\mathcal{C}$ degenerates as the line at infinity and the circumconic with perspector $X(1)$, and has parent $K202 = \mathrm{p}\mathcal{K}(X(1), X(144))$.

Example 17. The complement of $X(8)$, the Nagel point, is $X(1)$, the incenter, the anticomplement of $X(8)$ is $X(145)$. The center of the inconic with perspector $X(1)$ is $X(9)$. If we apply Theorem 19 with $X = X(8)$, then we get a cubic $\mathcal{C} = \mathrm{n}\mathcal{K}(X(9), X(145), k)$ with parent $\mathrm{p}\mathcal{K}(X(9), G)$. The Grassmann points are $X(8)$ and $X(1)$. The cubic $\mathrm{p}\mathcal{K}(X(9), G)$ does not appear in the current [2], but contains $X(1)$, $X(2)$, $X(8)$ and $X(9)$. The harmonic associate of $\mathcal{C}$ degenerates as the line at infinity and the circumconic with perspector $X(9)$, and has parent $K201 = \mathrm{p}\mathcal{K}(X(9), X(145))$.

Example 18. The complement of $X(69)$ is K, the anticomplement of $X(69)$ is $X(193)$. The center of the inconic with perspector $X(69)$ is O. If we apply Theorem 19 with $X = X(69)$, then we get a cubic $\mathcal{C} = \mathrm{n}\mathcal{K}(O, X(193), k)$ with parent $K168 = \mathrm{p}\mathcal{K}(O, G)$. The Grassmann points are $X(69)$ and K. The harmonic associate of $\mathcal{C}$ degenerates as the line at infinity and the circumconic with perspector O, and has parent $\mathrm{p}\mathcal{K}(O, X(193))$.

Example 19. The complement of $X(66)$ is $X(206)$, the anticomplement is not in [5], but appears in [2] as $P161$ - see $K161$. The center of the inconic with perspector $X(66)$ is $X(32)$. From Theorem 19 with $X = X(66)$, we get a cubic $\mathcal{C} = \mathrm{n}\mathcal{K}(X(32), P161, k)$ with parent $K177 = \mathrm{p}\mathcal{K}(X(32), G)$. The Grassmann points are $X(66)$ and $X(206)$. The harmonic associate of C degenerates as the line at infinity and the circumconic with perspector $X(32)$, and has parent $K161 = \mathrm{p}\mathcal{K}(X(32), P161)$.

In Examples 3 and 9, we met the kjp cubic $K024 = \mathrm{n}\mathcal{K}_0(K, K)$. The parent is the McCay cubic $K003 = \mathrm{p}\mathcal{K}(K, O)$. It follows that the harmonic associate of $K024$ is of the form $\mathrm{n}\mathcal{K}(K, O, k)$, and that this has parent the Grebe cubic $K102 = \mathrm{p}\mathcal{K}(K, K)$. From Theorem 9(b), the general $\mathrm{n}\mathcal{K}_0(W, W)$ has parent $\mathrm{p}\mathcal{K}(W, V)$, where V is the G-Ceva conjugate of W. The harmonic associate will be of the form $\mathrm{n}\mathcal{K}(W, V, k)$, with parent $\mathrm{p}\mathcal{K}(W, W)$. This means that the class $CL007$, which contains cubics $\mathrm{p}\mathcal{K}(W, W)$, is related to the class $CL009$, which contains cubics $\mathrm{p}\mathcal{K}(W, V)$, and to the class $CL026$, which contains cubics $\mathrm{n}\mathcal{K}_0(W, W)$. We give four examples. In general, the Grassmann points and cevian points may be complex.

Example 20. As above, the cubic $\mathrm{n}\mathcal{K}_0(X(1), X(1))$ has parent $\mathrm{p}\mathcal{K}(X(1), X(9))$. The harmonic associate $\mathrm{n}\mathcal{K}(X(1), X(9), k)$ has parent $K101 = \mathrm{p}\mathcal{K}(X(1), X(1))$.

Example 21. As above, the cubic $\mathrm{n}\mathcal{K}_0(H, H)$ has parent $K159 = \mathrm{p}\mathcal{K}(H, X(1249))$. The harmonic associate $\mathrm{n}\mathcal{K}(H, X(1249), k)$ has parent $K181 = \mathrm{p}\mathcal{K}(H, H)$.

Example 22. The cubic $\mathrm{n}\mathcal{K}_0(X(9), X(9))$ has parent $K157 = \mathrm{p}\mathcal{K}(X(9), X(1))$. The harmonic associate $\mathrm{n}\mathcal{K}(X(9), X(1), k)$ has parent $\mathrm{p}\mathcal{K}(X(9), X(9))$.

Example 23. The cubic $\mathrm{n}\mathcal{K}_0(X(32), X(32))$ has parent $K160 = \mathrm{p}\mathcal{K}(X(32), X(206))$. The harmonic associate $\mathrm{n}\mathcal{K}(X(32), X(206), k)$ has parent $\mathrm{p}\mathcal{K}(X(32), X(32))$.

8. Gibert's theorem

In private correspondence, Bernard Gibert has noted a further characterization of the vertices of the desmic structures we have used.

Theorem 21 (Gibert)**.** *Suppose that P,and Q are two W-isoconjugate points on the cubic $\mathrm{p}\mathcal{K}(W, S)$. For X on $\mathrm{p}\mathcal{K}(W, S)$, let X^t be the tangential of X, and $\mathrm{p}\mathcal{C}(X)$ be the polar conic of X. Now $\mathrm{p}\mathcal{C}(P^t)$ meets $\mathrm{p}\mathcal{K}(W, S)$ at P^t (twice), at P, and at three other points A', B', C', and $\mathrm{p}\mathcal{C}(Q^t)$ meets $\mathrm{p}\mathcal{K}(W, S)$ at Q^t (twice), at Q, and at three other points A'', B'', C''. Then the points A', B', C', A'', B'', C'' are the vertices of a desmic structure with perspectors P and Q.*

This can be verified computationally.

Appendix

We observe that the cubic $\mathrm{n}\mathcal{K}(W, R, k)$ meets the sidelines of $\triangle ABC$ at A, B, C and at the intersections with $\mathcal{T}(R)$. This accounts for all three intersections of the cubic with each sideline. The calculation referred to in Theorem 6(e) shows that $\mathrm{n}\mathcal{K}(W, R, k)$ and $\mathrm{p}\mathcal{K}(W, R_A)$ touch at B, C, meet at A, and at the intersections of $\mathcal{T}(R)$ with AB and AC. On algebraic grounds, there are nine intersections, so in the generic case, there are two further intersections. We now look at the Maple results in detail. If we look at the equations for $\mathrm{n}\mathcal{K}(W, R, k)$ and $\mathrm{p}\mathcal{K}(W, R_A)$ and solve for y, z, then we get the expected solutions, and $z = ax$, $y = \frac{-2vx(t+ar)}{2aus+k}$, where a satisfies

$$2u(2vtr - ks)X^2 + (4uvt^2 + 4vwr^2 - 4uvs^2 - k^2)X + 2w(2vtr - ks) = 0.$$

We cannot have $x = 0, ory = z = 0$. Thus we must have $a = 0$, giving $z = 0$, or $a = -\frac{t}{r}$, giving $y = 0$. The equation for a has a nonzero root only when $k = \frac{2vrt}{s}$. If we put $-\frac{t}{r}$ in the equation for a, we get $k = \frac{2ust}{r}$ or $k = \frac{2wrs}{t}$. If we replace $\mathrm{p}\mathcal{K}(W, R_A)$ by $\mathrm{p}\mathcal{K}(W, R_B)$ or $\mathrm{p}\mathcal{K}(W, R_C)$, we clearly get the same results. This establishes the criteria set out in Theorem 6(e).

If we consider the case $k = \frac{2vtr}{s}$, the equation for a becomes $X = 0$, so we can regard the solutions as limits as a tends to 0 or ∞. The former leads to $r : -s : 0$, the latter to $0 : 0 : 1$. We will meet these again below. We now examine the locus $GN(\triangle A'B'C')$ where the vertex $A' = a_1 : a_2 : a_3$ lies on a sideline. If $a_1 = 0$, the equation for the locus has some zero coefficients. This cannot include the cubic $\mathrm{n}\mathcal{K}(W, R, k)$ unless it is the whole plane. Thus we cannot define a cubic as a locus in this case.

Guided by the above discussion, we now examine the case $a_3 = 0$. Let $B' = b_1 : b_2 : b_3$, $C' = c_1 : c_2 : c_3$. The condition for the locus to be an $\mathrm{n}\mathcal{K}$ becomes $a_2b_3c_1 = 0$. Taking $a_2 = 0$ or $b_3 = 0$ leads to an equation with zero coefficients, so we must have $c_1 = 0$. When we equate the coefficients of the equations for the locus and $\mathrm{n}\mathcal{K}(W, R, k)$ other than y^2z and xyz, we find a unique solution. After scaling this is $A' = r : -s : 0$, $B' = u/r : -v/s : w/t$, $C' = 0 : -s : t$. Then the locus is $\mathrm{n}\mathcal{K}(W, R, 2vtr/s)$. A little thought shows that for fixed W, R, there are only three such loci, giving $\mathrm{n}\mathcal{K}(W, R, k)$ with $k = 2ust/r, 2vtr/s, 2wrs/t$. From our earlier work, we know that there is no other way to express these cubics as loci of type GN.

We should expect to obtain a desmic stucture by taking W-isoconjugates of A', B', C'. If we write the isoconjugate of $X = x : y : z$ as $uyz : vzx : wxy$, then the isconjugates are $A'' = C$, $B'' = r : -s : t = R_B$, $C'' = A$. Then $\triangle ABC$ and $\triangle A'B'C'$ have perspector B, $\triangle ABC$ and $\triangle CR_BA$ have perspector $P = r : 0 : t$, $\triangle CR_BA$ and $\triangle A'B'C'$ have perspector R_B.

It is a moot point whether this should be termed a desmic structure. It satisfies the perspectivity conditions, but has only eight distinct points. If we replace B' by any point, we still get the same perspectors. If we allow this as a desmic, then Theorem 7 holds as stated. If not, we can either add these three cubics to the excluded list or reword in the weaker form.

Theorem 7′. *If a cubic $\mathcal{C}$ is of type $\mathrm{n}\mathcal{K}$, but not of type $\mathrm{c}\mathcal{K}$, then there is a triangle Δ with $\mathcal{C} = GN(\Delta)$, and at most two such triangles.*

To get R as the barycentric difference of perspectors, we need to scale B to $0 : -s : 0$. Then the sum is R_B. A check using Theorem 9(b) shows that the parent is indeed $\mathrm{p}\mathcal{K}(W, R_B) = GP(\triangle A'B'C')$. Replacing each coordinate of P in turn by the corresponding one from B, we get C, R_B, A, as expected. On the other hand, starting from B, we get A', $0 : 0 : 0$, C'. This reflects the fact that the other points do not determine B'. When we compute the equation of $GN(\triangle CR_BA)$, we find that all the coefficients are zero. Thus cubics of this kind are Grassmann cubics for only one triangle rather than the usual two.

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