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The Orthic-of-Intouch and Intouch-of-Orthic Triangles

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Abstract. Barycentric coordinates are used to prove that the othic of intouch and intouch of orthic triangles are homothetic. Indeed, both triangles are homothetic to the reference triangle. Ratios and centers of homothety are found, and certain collinearities are proved.

1. Introduction

We consider a pair of triangles associated with a given triangle: the orthic triangle of the intouch triangle, and the intouch triangle of the orthic triangle. See Figure 1. Clark Kimberling [1, p. 274] asks if these two triangles are homothetic. We shall show that this is true if the given triangle is acute, and indeed each of them is homothetic to the reference triangle. In this paper, we adopt standard notations of triangle geometry, and denote the side lengths of triangle ABC by a, b, c. Let I denote the incenter, and the incircle (with inradius r) touching the sidelines BC, CA, AB at D, E, F respectively, so that DEF is the intouch triangle of ABC. Let H be the orthocenter of ABC, and let

$$D' = AH \cap BC, \quad E' = BH \cap CA, \quad F' = CH \cap AB,$$

so that $D'E'F'$ is the orthic triangle of ABC. We shall also denote by O the circumcenter of ABC and R the circumradius. In this paper we make use of homogeneous barycentric coordinates. Here are the coordinates of some basic triangle centers in the notations introduced by John H. Conway:

$$I =(a : b : c), \qquad H = \left(\frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C}\right) = (S_{BC} : S_{CA} : S_{AB}),$$
$$O =(a^2 S_A : b^2 S_B : c^2 S_C) = (S_A(S_B + S_C) : S_B(S_C + S_A) : S_C(S_A + S_B)),$$

where

$$S_A = \frac{b^2 + c^2 - a^2}{2}, \quad S_B = \frac{c^2 + a^2 - b^2}{2}, \quad S_C = \frac{a^2 + b^2 - c^2}{2},$$

and

$$S_{BC} = S_B \cdot S_C, \qquad S_{CA} = S_C \cdot S_A, \qquad S_{AB} = S_A \cdot S_B.$$

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2. Two pairs of homothetic triangles

2.1. *Perspectivity of a cevian triangle and an anticevian triangle.* Let P and Q be arbitrary points not on any of the sidelines of triangle ABC. It is well known that the cevian triangle of $P = (u : v : w)$ is perspective with the anticevian triangle of $Q = (x : y : z)$ at

$$P/Q = \left(x\left(-\frac{x}{u}+\frac{y}{v}+\frac{z}{w}\right) : y\left(\frac{x}{u}-\frac{y}{v}+\frac{z}{w}\right) : z\left(\frac{x}{u}+\frac{y}{v}-\frac{z}{w}\right)\right).$$

See, for example, [3, §8.3].

2.2. *The intouch and the excentral triangles.* The intouch and the excentral triangles are homothetic since their corresponding sides are perpendicular to the respective angle bisectors of triangle ABC. The homothetic center is the point

$$\begin{aligned} P_1 =&(a(-a(s-a)+b(s-b)+c(s-c)) : b(a(s-a)-b(s-b)+c(s-c)) \\ &: c(a(s-a)+b(s-b)-c(s-c))) \\ =&(a(s-b)(s-c) : b(s-c)(s-a) : c(s-a)(s-b)) \\ =&\left(\frac{a}{s-a} : \frac{b}{s-b} : \frac{c}{s-c}\right). \end{aligned}$$

This is the triangle center X_{57} in [2].

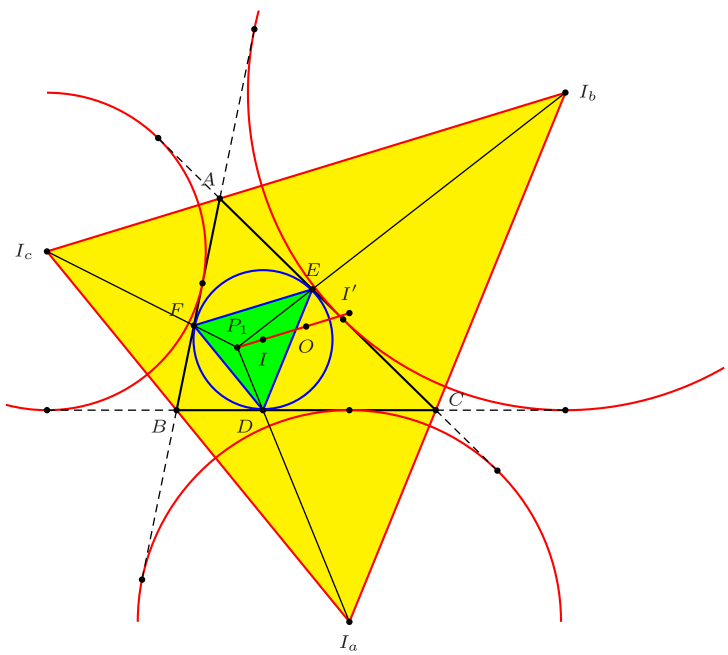

Figure 1.

2.3. *The orthic and the tangential triangle.* The orthic triangle and the tangential triangle are also homothetic since their corresponding sides are perpendicular to the respective circumradii of triangle ABC. The homothetic center is the point

$$\begin{aligned}P_2 =&\left(a^2(-a^2S_A+b^2S_B+c^2S_C):b^2(-b^2S_B+c^2S_C+a^2S_A)\right.\\&\left.:c^2(-c^2S_C+a^2S_A+b^2S_B)\right)\\=&(a^2S_{BC}:b^2S_{CA}:c^2S_{AB})\\=&\left(\frac{a^2}{S_A}:\frac{b^2}{S_B}:\frac{c^2}{S_C}\right).\end{aligned}$$

This is the triangle center X_{25} in [2].

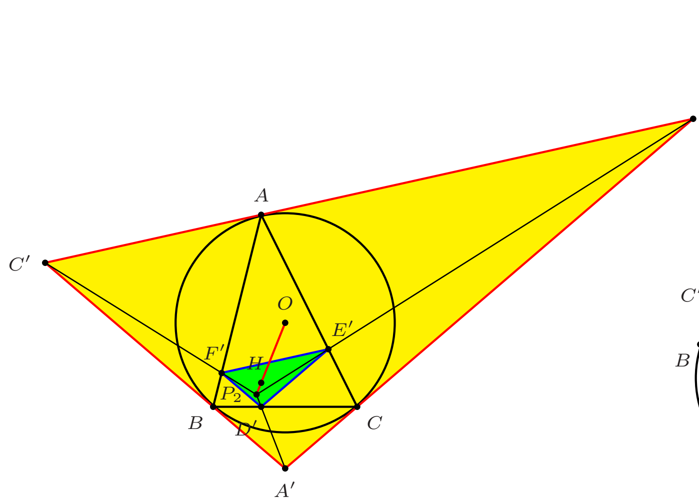

Figure 2A.

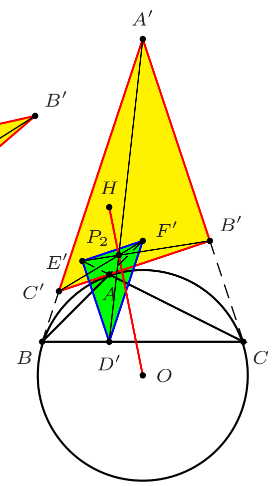

Figure 2B.

The ratio of homothety is positive or negative according as ABC is acute-angled and obtuse-angled. [1] See Figures 2A and 2B. When ABC is acute-angled, HD', HE' and HF' are the angle bisectors of the orthic triangle, and H is the incenter of the orthic triangle. If ABC is obtuse-angled, the incenter of the orthic triangle is the obtuse angle vertex.

3. The orthic-of-intouch triangle

The orthic-of-intouch triangle of ABC is the orthic triangle UVW of the intouch triangle DEF. Let h_1 be the homothety with center P_1, swapping D, E, F into U, V, W respectively. Consider an altitude DU of DEF. This is the image of the altitude I_aA of the excentral triangle under the homothety h_1. In particular, $U = \mathsf{h}_1(A)$. See Figure 3. Similarly, the same homothety maps B and C

[1] This ratio of homothety is $2\cos A\cos B\cos C$.

into V and W respectively. It follows that UVW is the image of ABC under the homothety h_1.

Since the circumcircle of UVW is the nine-point circle of DEF, it has radius $\frac{r}{2}$. It follows that the ratio of homothety is $\frac{r}{2R}$.

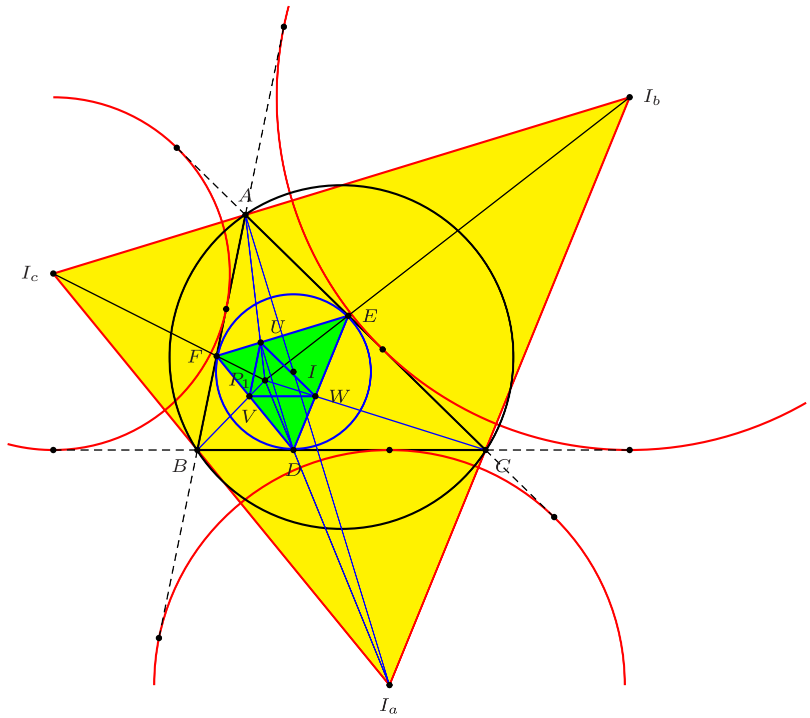

Figure 3.

Proposition 1. *The vertices of the orthic-of-intouch triangle are*

$$U =((b+c)(s-b)(s-c) : b(s-c)(s-a) : c(s-a)(s-b)) = \left(\frac{b+c}{s-a} : \frac{b}{s-b} : \frac{c}{s-c}\right),$$
$$V =(a(s-b)(s-c) : (c+a)(s-c)(s-a) : c(s-a)(s-b)) = \left(\frac{a}{s-a} : \frac{c+a}{s-b} : \frac{c}{s-c}\right),$$
$$W =(a(s-b)(s-c) : b(s-c)(s-a) : (a+b)(s-a)(s-b)) = \left(\frac{a}{s-a} : \frac{b}{s-b} : \frac{a+b}{s-c}\right).$$

Proof. The intouch triangle DEF has vertices

$$D = (0 : s-c : s-b), \qquad E = (s-c : 0 : s-a), \qquad F = (s-b : s-a : 0).$$

The sidelines of the intouch triangle have equations

$$\begin{aligned} EF: &\quad -(s-a)x + (s-b)y + (s-c)z = 0,\\ FD: &\quad (s-a)x - (s-b)y + (s-c)z = 0,\\ DE: &\quad (s-a)x + (s-b)y - (s-c)z = 0. \end{aligned}$$

The point U is the intersection of the lines AP_1 and EF. See Figure 3. The line AP_1 has equation

$$-c(s-b)y+b(s-c)z=0.$$

Solving this with that of EF, we obtain the coordinates of U given above. Those of V and W are computed similarly. □

Corollary 2. *The equations of the sidelines of the orthic-of-intouch triangle are*

$$\begin{aligned} VW:&\quad -s(s-a)x+(s-b)(s-c)y+(s-b)(s-c)z=0,\\ WU:&\quad (s-c)(s-a)x-s(s-b)y+(s-c)(s-a)z=0,\\ UV:&\quad (s-a)(s-b)x+(s-a)(s-b)y-s(s-c)z=0. \end{aligned}$$

4. The intouch-of-orthic triangle

Suppose triangle ABC is acute-angled, so that its orthic triangle $D'E'F'$ has incenter H, and is the image of the tangential triangle $A'B'C'$ under a homothety h_2 with center P_2. Consider the intouch triangle XYZ of $D'E'F'$. Under the homothety h_2, the segment $A'A$ is swapped into $D'X$. See Figure 4. In particular, $\mathsf{h}_2(A)=X$. For the same reason, $\mathsf{h}_2(B)=Y$ and $\mathsf{h}_2(C)=Z$. Therefore, the intouch-of-orthic triangle XYZ is homothetic to ABC under h_2.

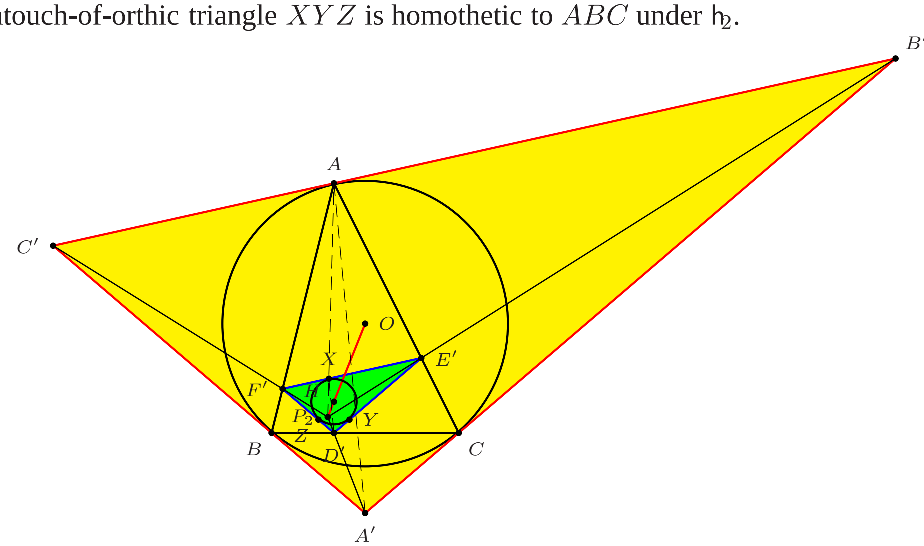

Figure 4

Proposition 3. *If ABC is acute angled, the vertices of the intouch-of-orthic triangle are*

$$\begin{aligned} X =&((b^2+c^2)S_{BC}:b^2S_{CA}:c^2S_{AB})=\left(\frac{b^2+c^2}{S_A}:\frac{b^2}{S_B}:\frac{c^2}{S_C}\right),\\ Y =&(a^2S_{BC}:(c^2+a^2)S_{CA}:c^2S_{AB})=\left(\frac{a^2}{S_A}:\frac{c^2+a^2}{S_B}:\frac{c^2}{S_C}\right),\\ Z =&(a^2S_{BC}:b^2S_{CA}:(a^2+b^2)S_{AB})=\left(\frac{a^2}{S_A}:\frac{b^2}{S_B}:\frac{a^2+b^2}{S_C}\right). \end{aligned}$$

Proof. The orthic triangle $D'E'F'$ has vertices

$$D' = (0 : S_C : S_B), \qquad E' = (S_C : 0 : S_A), \qquad F' = (S_B : S_A : 0).$$

The sidelines of the orthic triangle have equations

$$\begin{array}{ll} E'F': & -S_A x + S_B y + S_C z = 0, \\ F'D': & \quad S_A x - S_B y + S_C z = 0, \\ D'E': & \quad S_A x + S_B y - S_C z = 0. \end{array}$$

The point X is the intersection of the lines AP_2 and $E'F'$. See Figure 4. The line AP_2 has equation

$$-c^2 S_B y + b^2 S_C z = 0.$$

Solving this with that of $E'F'$, we obtain the coordinates of U given above. Those of Y and Z are computed similarly. □

Corollary 4. *If ABC is acute-angled, the equations of the sidelines of the intouch-of-orthic triangle are*

$$\begin{array}{ll} YZ: & -S_A(S_A + S_B + S_C)x + S_{BC}y + S_{BC}z = 0, \\ ZX: & S_{CA}x - S_B(S_A + S_B + S_C)y + S_{CA}z = 0, \\ UV: & S_{AB}x + S_{AB}y - S_C(S_A + S_B + S_C)z = 0. \end{array}$$

5. Homothety of the intouch-of-orthic and orthic-of-intouch triangles

Proposition 5. *If triangle ABC is acute angled, then its intouch-of-orthic and orthic-of-intouch triangles are homothetic at the point*

$$Q = \left(\frac{a(a(b+c) - (b^2 + c^2))}{(s-a)S_A} : \frac{b(b(c+a) - (c^2 + a^2))}{(s-b)S_B} : \frac{c(c(a+b) - (a^2 + b^2))}{(s-c)S_C} \right).$$

Proof. The homothetic center is the intersection of the lines UX, VY, and WZ. See Figure 5. Making use of the coordinates given in Propositions 1 and 3, we obtain the equations of these lines as follows.

$$\begin{array}{ll} UX: & \quad bc(s-a)S_A(c(s-c)S_B - b(s-b)S_C)x \\ & +c(s-b)S_B((b^2+c^2)(s-a)S_C - (b+c)c(s-c)S_A)y \\ & +b(s-c)S_C(b(b+c)(s-b)S_A - (b^2+c^2)(s-a)S_B)z = 0, \\ VY: & \quad c(s-a)S_A(c(c+a)(s-c)S_B - (c^2+a^2)(s-b)S_C)x \\ & +ca(s-b)S_B(a(s-a)S_C - c(s-c)S_A)y \\ & +a(s-c)S_C((c^2+a^2)(s-b)S_A - (c+a)a(s-a)S_B)z = 0, \\ WZ: & \quad b(s-a)S_A((a^2+b^2)(s-c)S_B - (a+b)b(s-b)S_C)x \\ & +a(s-b)S_B(a(a+b)(s-a)S_C - (a^2+b^2)(s-c)S_A)y \\ & +ab(s-c)S_C(b(s-b)S_A - a(s-a)S_B)z = 0. \end{array}$$

It is routine to verify that Q lies on each of these lines. □

Remark. Q is the triangle center X_{1876} in [2].

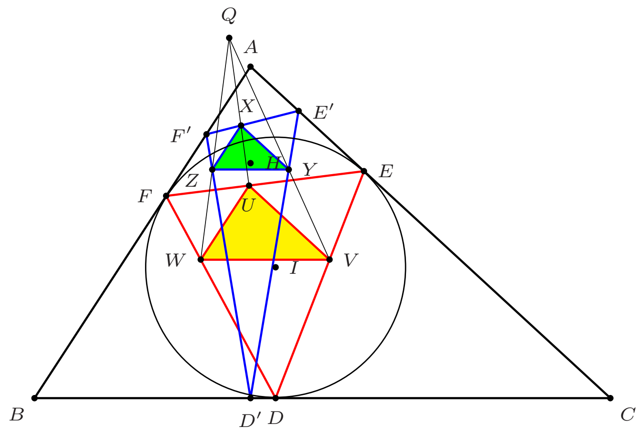

Figure 5

6. Collinearities

Because the circumcenter of XYZ is the orthocenter H of ABC, the center of homothety P_2 of ABC and XYZ lies on the Euler line OH of ABC. See Figure 4. We demonstrate a similar property for the point P_1, namely, that this point lies on the Euler line IF of DEF, where F is the circumcenter of UVW. Clearly, O, F, P_1 are collinear. Therefore, it suffices to prove that the points I, O, P_1 are collinear. This follows from

$$\begin{vmatrix} 1 & 1 & 1 \\ \cos A & \cos B & \cos C \\ (s-b)(s-c) & (s-c)(s-a) & (s-a)(s-b) \end{vmatrix} = 0,$$

which is quite easy to check. See Figure 1.

References

[1] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.

[2] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[3] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University lecture notes, 2001.

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