Forum Geometricorum
Volume 6 (2006) 199–203.

FORUM GEOM
ISSN 1534-1178

A Simple Perspectivity

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Abstract. We construct a simple perspectivity that is invariant under isotomic conjugation.

1. Introduction

In this note we consider a simple transformation of the plane of a given reference triangle ABC. Given a point P with cevian triangle XYZ, construct the parallels through B to XY and through C to XZ to intersect at A'; similarly define B' and C'. Construct

$$A^* = BB' \cap CC', \qquad B^* = CC' \cap AA', \qquad C^* = AA' \cap BB'.$$

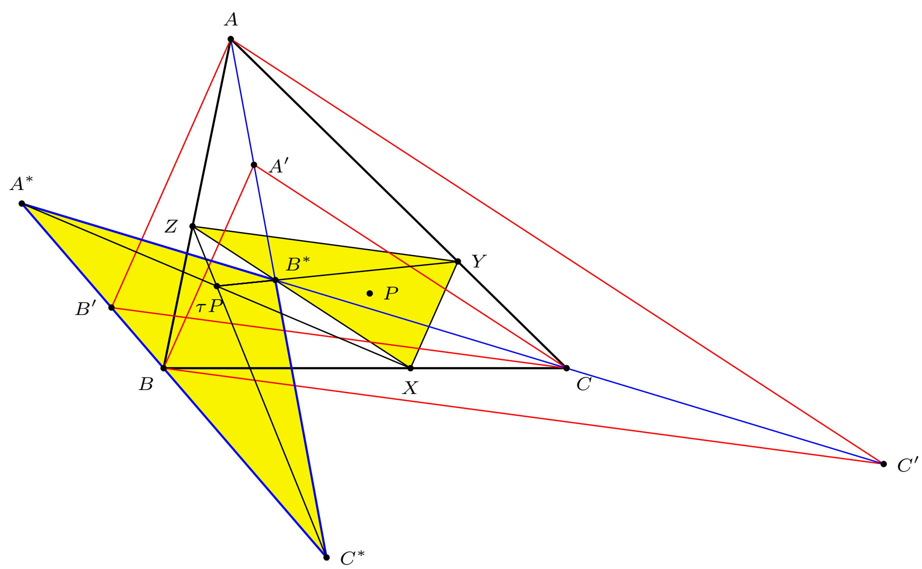

Figure 1

Proposition 1. *Triangle $A^*B^*C^*$ is the anticevian triangle of the infinite point $Q = (u(v-w) : v(w-u) : w(u-v))$ of the trilinear polar of P.*

We shall prove Proposition 1 in §2 below. As an anticevian triangle, $A^*B^*C^*$ is perspective with every cevian triangle. In particular, it is perspective with XYZ at the cevian quotient P/Q, which depends on P only. We write

$$\tau(P) := P/Q = (u(v-w)^2 : v(w-u)^2 : w(u-v)^2).$$

Publication Date: June 12, 2006. Communicating Editor: Paul Yiu.
The author thanks Paul Yiu for his help in the preparation of this paper.

Let $P^\bullet$ denote the isotomic conjugate of P.

Proposition 2. *τ is invariant under isotomic conjugation: $\tau(P^\bullet) = \tau(P)$.*

Proposition 3. *$\tau(P)$ is*
(1) *the center of the circumconic through P and its isotomic conjugate $P^\bullet$,*
(2) *the perspector of the circum-hyperbola with asymptotes the trilinear polars of P and $P^\bullet$.*

Proof. (1) The circumconic through P and $P^\bullet$ has equation

$$\frac{u(v^2-w^2)}{x} + \frac{v(w^2-u^2)}{y} + \frac{w(u^2-v^2)}{z} = 0,$$

with perspector

$$P' = (u(v^2-w^2) : \ w(w^2-u^2) : \ w(u^2-v^2)). \tag{1}$$

Its center is the cevian quotient G/P'. This is $\tau(P)$.

(2) The pencil of hyperbolas with asymptotes the trilinears polars of P and $P^\bullet$ has equation

$$k(x+y+z)^2 + (ux+vy+cz)\left(\frac{x}{u} + \frac{y}{v} + \frac{z}{w}\right) = 0.$$

For $k = -1$, the hyperbola passes through A, B, C, and this circum-hyperbola has equation

$$\frac{u(v-w)^2}{x} + \frac{v(w-u)^2}{y} + \frac{w(u-v)^2}{z} = 0.$$

It has perspector $\tau(P)$, (and center P' given in (1) above). □

Remark. Wilson Stothers [2] has found that one asymptote of a circum-hyperbola determines the other. More precisely, if $ux + vy + wz = 0$ is an asymptote of a circum-hyperbola, then the other is $\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0$. This gives a stronger result than (2) above.

Here is a list of triangle centers with their images under τ. The labeling of triangle centers follows Kimberling [1].

P, $P^\bullet$	$\tau(P)$		P, $P^\bullet$	$\tau(P)$
X_1, X_{75}	X_{244}		$X_3,\ X_{264}$	X_{2972}
$X_4,\ X_{69}$	X_{125}		$X_7,\ X_8$	X_{11}
$X_{20},\ X_{253}$	X_{122}		$X_{30},\ X_{1494}$	X_{1650}
$X_{57},\ X_{312}$	X_{2170}		$X_{88},\ X_{88}^\bullet$	X_{2087}
$X_{94},\ X_{323}$	X_{2088}		$X_{98},\ X_{325}$	X_{868}
$X_{99},\ X_{523}$	X_{1649}		$X_{200},\ X_{1088}$	X_{2310}
$X_{519},\ X_{903}$	X_{1647}		$X_{524},\ X_{671}$	X_{1648}
$X_{536},\ X_{536}^\bullet$	X_{1646}		$X_{538},\ X_{538}^\bullet$	X_{1645}
$X_{694},\ X_{694}^\bullet$	X_{2086}		$X_{1022},\ X_{1022}^\bullet$	X_{1635}
$X_{1026},\ X_{1026}^\bullet$	X_{2254}		$X_{2394},\ X_{2407}$	X_{1637}
$X_{2395},\ X_{2396}$	X_{2491}		$X_{2398},\ X_{2400}$	X_{676}

2. Proof of Proposition 1

The line XY has equation $vwx+wuy-uvz = 0$, and infinite point $(-u(v+w) : v(w + u) : w(u - v))$. The parallel through B to XY is the line

$$w(u - v)x + u(v + w)z = 0.$$

Similarly, the parallel through C to XZ is the line

$$v(u - w)x + u(v + w)y = 0.$$

These two lines intersect at

$$A' = (u(v + w) : v(w - u) : w(v - u)).$$

The two analogously defined points are

$$\begin{aligned} B' =&(u(w - v) : v(w + u) : w(u - v)),\\ C' =&(u(v - w) : v(u - w) : w(u + v)). \end{aligned}$$

Now the lines AA', BB', CC' intersect at the points

$$\begin{aligned} A^* =&BB' \cap CC' = (-u(v - w) : v(w - u) : w(u - v)),\\ B^* =&CC' \cap AA' = (u(v - w) : -v(w - u) : w(u - v)),\\ C^* =&AA' \cap BB' = (u(v - w) : v(w - u) : -w(u - v)). \end{aligned}$$

This is clearly the anticevian triangle of the point

$$Q = (u(v - w) : v(w - u) : w(u - v)) = \left(\frac{1}{v} - \frac{1}{w} : \frac{1}{w} - \frac{1}{u} : \frac{1}{u} - \frac{1}{v}\right),$$

which is the infinite point of the trilinear polar $\mathcal{L}$. This completes the proof of Proposition 1.

Remarks. (1) Here is an easy alternative construction of $A^*B^*C^*$. Construct the parallels through A, B, C to the trilinear polar $\mathcal{L}$, intersecting the sidelines BC, CA, AB at A_1, B_1, C_1 respectively. Then, A^*, B^*, C^* are the midpoints of the segments AA_1, BB_1, CC_1. See Figure 2.

(2) The equations of the sidelines of triangle $A^*B^*C^*$ are

$$\begin{aligned} B^*C^* &: \frac{y}{v(w-u)} + \frac{z}{w(u-v)} = 0,\\ C^*A^* &: \frac{x}{u(v-w)} + \frac{z}{w(u-v)} = 0,\\ A^*B^* &: \frac{x}{u(v-w)} + \frac{y}{v(w-u)} = 0. \end{aligned}$$

Proposition 4. *The trilinear polar of $\tau(P)$ with respect to the cevian triangle of P passes through P.*

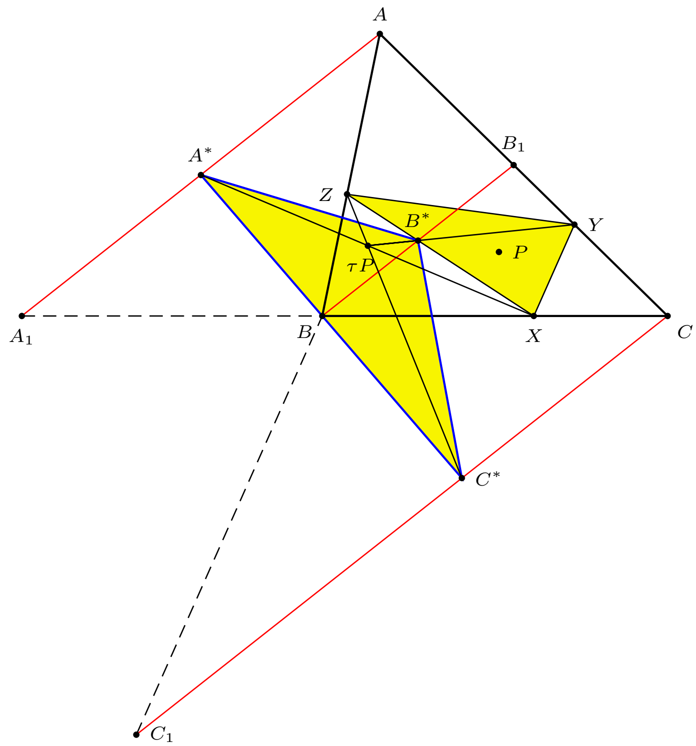

Figure 2

Proof. The trilinear polar of $\tau(P)$ with respect to XYZ is the perspectrix of the triangles XYZ and $A^*B^*C^*$. Now, the sidelines of these triangle intersect at the points

$$\begin{aligned} B^*C^* \cap YZ =& (u(v+w-2u) : \ v(w-u) : \ -w(u-v)), \\ C^*A^* \cap ZX =& (-u(v-w) : \ v(w+u-2v) : \ w(u-v)), \\ A^*B^* \cap XY =& (u(v-w) : \ -v(w-u) : \ w(u+v-2w)). \end{aligned}$$

The line through these three points has equation

$$\frac{v-w}{u}x + \frac{w-u}{v}y + \frac{u-v}{w}z = 0.$$

This clearly contains the point $P = (u : v : w)$. □

3. Generalization

Since the construction in §1 is purely perspective we can replace the line at infinity by an arbitrary line $\ell : px + qy + rz = 0$. The parallel through B to XY becomes the line joining B to the intersection ℓ and XY, etc. The perspector becomes

$$\tau_\ell(P) = (u(qv - rw)^2 : \ v(rw - pu)^2 : \ w(pu - qv)^2).$$

Then $\tau_\ell(P^\ell) = \tau(P)$ where $P^\ell = \left(\frac{1}{p^2u} : \frac{1}{q^2v} : \frac{1}{r^2w}\right)$, and the following remain valid:

(1) $A^*B^*C^*$ is the anticevian triangle of $Q = \mathcal{L} \cap \ell$, where $\mathcal{L}$ is the trilinear polar of P.
(2) The perspectrix of XYZ and $A^*B^*C^*$ contains the point P.

References

[1] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.
[2] W. Stothers, Hyacinthos message 8476, October 30, 2003.

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