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Simmons Conics

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Abstract. We study the conics introduced by T. C. Simmons and generalize some of their properties.

1. Introduction

In [1, Tome 3, p.227], we find a definition of a conic called "ellipse de Simmons" with a reference to E. Vigarié series of papers [8] in 1887-1889. According to Vigarié, this "ellipse" was introduced by T. C. Simmons [7], and has foci the first isogonic center or Fermat point (X_{13} in [5]) and the first isodynamic point (X_{15} in [5]). The contacts of this "ellipse" with the sidelines of reference triangle ABC are the vertices of the cevian triangle of X_{13}. In other words, the perspector of this conic is one of its foci. The given trilinear equation is :

$$\sqrt{\alpha \sin\left(A+\frac{\pi}{3}\right)}+\sqrt{\beta \sin(\left(B+\frac{\pi}{3}\right)}+\sqrt{\gamma \sin\left(C+\frac{\pi}{3}\right)}=0.$$

It appears that this conic is not always an ellipse and, curiously, the corresponding conic with the other Fermat and isodynamic points is not mentioned in [1].

In this paper, working with barycentric coordinates, we generalize the study of inscribed conics and circumconics whose perspector is one focus.

2. Circumconics and inscribed conics

Let $P = (u : v : w)$ be any point in the plane of triangle ABC which does not lie on one sideline of ABC. Denote by $\mathcal{L}(P)$ its trilinear polar.

The locus of the trilinear pole of a line passing through P is a circumconic denoted by $\Gamma_c(P)$ and the envelope of trilinear polar of points of $\mathcal{L}(P)$ is an inscribed conic $\Gamma_i(P)$. In both cases, P is said to be the perspector of the conic and $\mathcal{L}(P)$ its perspectrix. Note that $\mathcal{L}(P)$ is the polar line of P in both conics.

The centers of $\Gamma_c(P)$ and $\Gamma_i(P)$ are

$$\Omega_c(P) =(u(v+w-u)\ :v(w+u-v)\ :w(u+v-w)),$$
$$\Omega_i(P) =(u(v+w)\ :v(w+u)\ :w(u+v))$$

respectively. $\Omega_c(P)$ is also the perspector of the medial triangle and the anticevian triangle $A_PB_PC_P$ of P. $\Omega_i(P)$ is the complement of the isotomic conjugate of P.

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2.1. *Construction of the axes of* $\Gamma_c(P)$ *and* $\Gamma_i(P)$. Let X be the fourth intersection of the conic and the circumcircle (X is the trilinear pole of the line KP). The axes of $\Gamma_c(P)$ are the parallels at $\Omega_c(P)$ to the bisectors of the lines BC and AX. A similar construction in the cevian triangle $P_aP_bP_c$ of P gives the axes of $\Gamma_i(P)$.

2.2. *Construction of the foci of* $\Gamma_c(P)$ *and* $\Gamma_i(P)$. The line BC and its perpendicular at P_a meet one axis at two points. The circle with center $\Omega_i(P)$ which is orthogonal to the circle having diameter these two points meets the axis at the requested foci. A similar construction in the anticevian triangle of P gives the foci of $\Gamma_c(P)$.

3. Inscribed conics with focus at the perspector

Theorem 1. *There are two and only two non-degenerate inscribed conics whose perspector* P *is one focus : they are obtained when* P *is one of the isogonic centers.*

Proof. If P is one focus of $\Gamma_i(P)$, the other focus is the isogonal conjugate P^* of P and the center is the midpoint of PP^*. This center must be the isotomic conjugate of the anticomplement of P. A computation shows that P must lie on three circum-strophoids with singularity at one vertex of ABC. These strophoids are orthopivotal cubics as seen in [4, p.17]. They are the isogonal transforms of the three Apollonian circles which intersect at the two isodynamic points. Hence, the strophoids intersect at the isogonic centers. □

These conics will be called the (inscribed) *Simmons conics* denoted by $\mathcal{S}_{13} = \Gamma_i(X_{13})$ and $\mathcal{S}_{14} = \Gamma_i(X_{14})$.

Elements of the conics	$\mathcal{S}_{13}$	$\mathcal{S}_{14}$
perspector and focus	X_{13}	X_{14}
other real focus	X_{15}	X_{16}
center	X_{396}	X_{395}
focal axis	parallel to the Euler line	idem
non-focal axis	$\mathcal{L}(X_{14})$	$\mathcal{L}(X_{13})$
directrix	$\mathcal{L}(X_{13})$	$\mathcal{L}(X_{14})$
other directrix	$\mathcal{L}(X_{18})$	$\mathcal{L}(X_{17})$

Remark. The directrix associated to the perspector/focus in both Simmons conics is also the trilinear polar of this same perspector/focus. This will be generalized below.

Theorem 2. *The two (inscribed) Simmons conics generate a pencil of conics which contains the nine-point circle.*

The four (not always real) base points of the pencil form a quadrilateral inscribed in the nine point circle and whose diagonal triangle is the anticevian triangle of X_{523}, the infinite point of the perpendiculars to the Euler line. In Figure 1 we have four real base points on the nine point circle and on two parabolas $\mathcal{P}_1$ and $\mathcal{P}_2$.

Hence, all the conics of the pencil have axes with the same directions (parallel and perpendicular to the Euler line) and are centered on the rectangular hyperbola

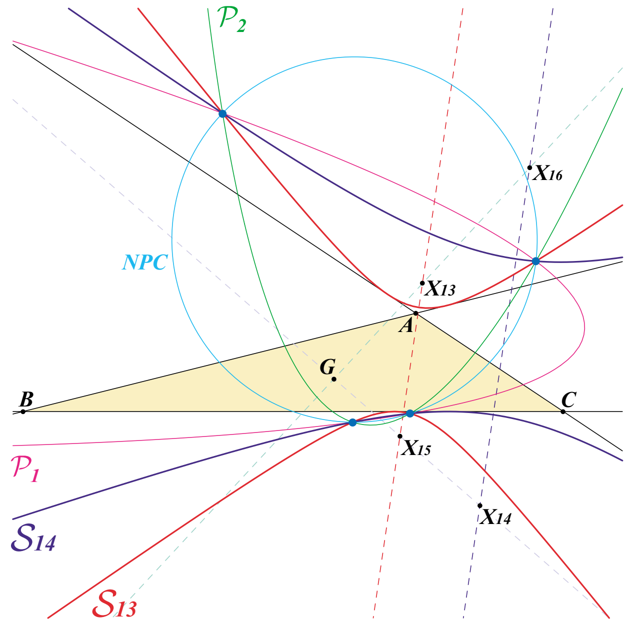

Figure 1. Simmons ponctual pencil of conics

which is the polar conic of X_{30} (point at infinity of the Euler line) in the Neuberg cubic. This hyperbola passes through the in/excenters, X_5, X_{30}, X_{395}, X_{396}, X_{523}, X_{1749} and is centered at X_{476} (Tixier point). See Figure 2. This is the diagonal conic with equation :

$$\sum_{\text{cyclic}} (b^2 - c^2)(4S_A^2 - b^2c^2)x^2 = 0.$$

It must also contain the vertices of the anticevian triangle of any of its points and, in particular, those of the diagonal triangle above. Note that the polar lines of any of its points in both Simmons inconics are parallel.

Theorem 3. *The two (inscribed) Simmons conics generate a tangential pencil of conics which contains the Steiner inellipse.*

Indeed, their centers X_{396} and X_{395} lie on the line GK. The locus of foci of all inconics with center on this line is the (second) Brocard cubic K018 which is $\mathrm{n}\mathcal{K}_0(K, X_{523})$ (See [3]). These conics must be tangent to the trilinear polar of the root X_{523} which is the line through the centers X_{115} and X_{125} of the Kiepert and Jerabek hyperbolas.

Another approach is the following. The fourth common tangent to two inconics is the trilinear polar of the intersection of the trilinear polars of the two perspectors. In the case of the Simmons inconics, the intersection is X_{523} at infinity (the perspector of the Kiepert hyperbola) hence the common tangent must be the trilinear polar of this point. In fact, more generally, any inconic with perspector on the

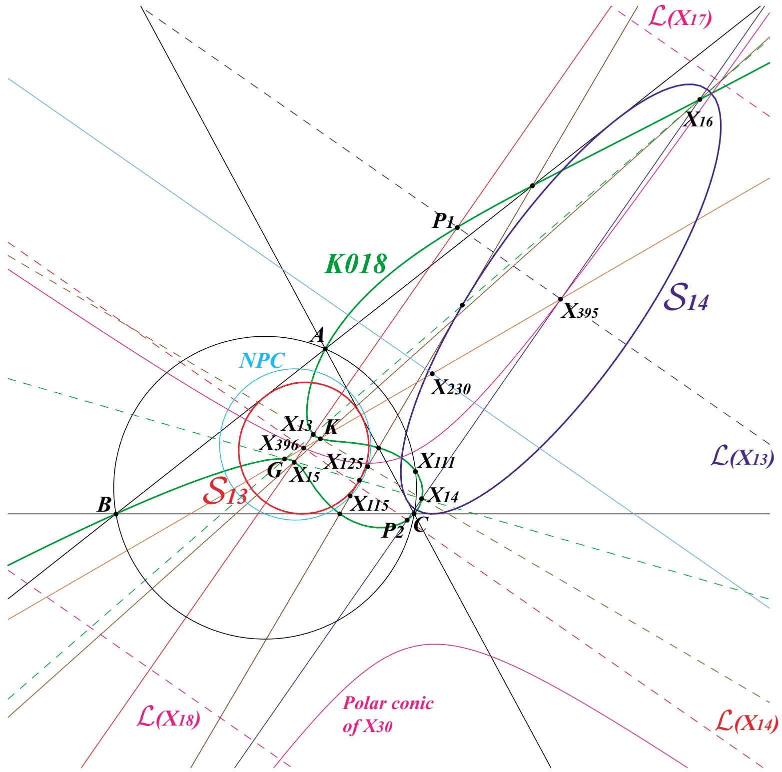

Figure 2. The two Simmons inconics $\mathcal{S}_{13}$ and $\mathcal{S}_{14}$

Kiepert hyperbola must be tangent to this same line (the perpector of each conic must lie on the Kiepert hyperbola since it is the isotomic conjugate of the anticomplement of the center of the conic). In particular, since G lies on the Kiepert hyperbola, the Steiner inellipse must also be tangent to this line. This is also the case of the inconic with center K, perspector H sometimes called K-ellipse (see [1]) although it is not always an ellipse.

Remarks. (1) The contacts of this common tangent with $\mathcal{S}_{13}$ and $\mathcal{S}_{14}$ lie on the lines through G and the corresponding perspector. See Figures 2 and 3.

(2) This line $X_{115}X_{125}$ meets the sidelines of ABC at three points on K018.

(3) The focal axes meet the non-focal axes at the vertices of a rectangle with center X_{230} on the orthic axis and on the line GK. These vertices are X_{396}, X_{395} and two other points P_1, P_2 on the cubic K018 and collinear with X_{111}, the singular focus of the cubic.

(4) The orthic axis is the mediator of the non-focal axes.

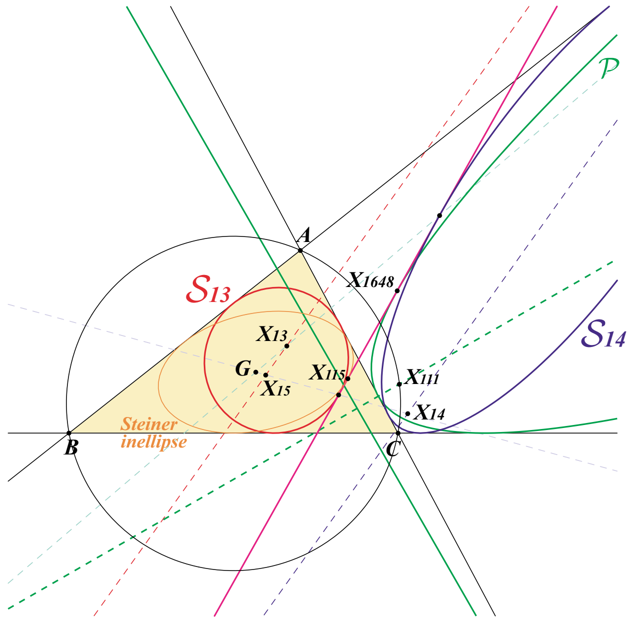

Figure 3. Simmons tangential pencil of conics

(5) The pencil contains one and only one parabola $\mathcal{P}$ we will call *the Simmons parabola*. This is the in-parabola with perspector X_{671} (on the Steiner ellipse), focus X_{111} (Parry point), touching the line $X_{115}X_{125}$ at X_{1648}.[1]

4. Circumconics with focus at the perspector

A circumconic with perspector P is inscribed in the anticevian triangle $P_aP_bP_c$ of P. In other words, it is the inconic with perspector P in $P_aP_bP_c$. Thus, P is a focus of the circumconic if and only if it is a Fermat point of $P_aP_bP_c$. According to a known result [2], it must then be a Fermat point of ABC. Hence,

Theorem 4. *There are two and only two non-degenerate circumconics whose perspector P is one focus : they are obtained when P is one of the isogonic centers.*

They will be called the *Simmons circumconics* denoted by $\Sigma_{13} = \Gamma_c(X_{13})$ and $\Sigma_{14} = \Gamma_c(X_{14})$. See Figure 4.

The fourth common point of these conics is X_{476} (Tixier point) on the circumcircle. The centers and other real foci are not mentioned in the current edition of [6] and their coordinates are rather complicated. The focal axes are those of the Simmons inconics.

[1] X_{1648} is the tripolar centroid of X_{523} i.e. the isobarycenter of the traces of the line $X_{115}X_{125}$. It lies on the line GK.

[2] The angular coordinates of a Fermat point of $P_aP_bP_c$ are the same when they are taken either with respect to $P_aP_bP_c$ or with respect to ABC.

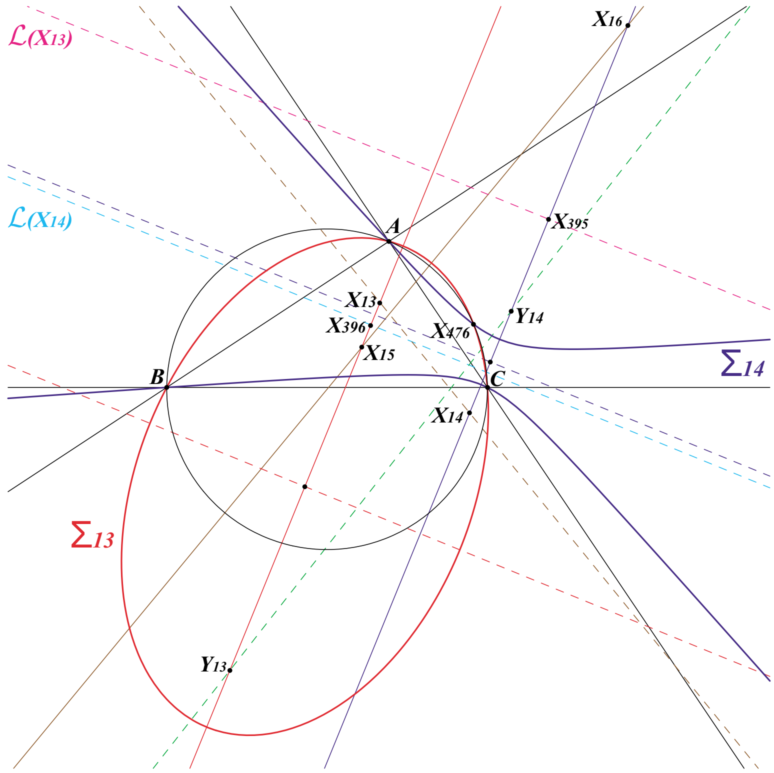

Figure 4. The two Simmons circumconics Σ_{13} and Σ_{14}

A digression: there are in general four circumconics with given focus F. Let $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$ the circles passing through F with centers A, B, C. These circles have two by two six centers of homothety and these centers are three by three collinear on four lines. One of these lines is the trilinear polar $\mathcal{L}(Q)$ of the interior point $Q = \frac{1}{AF} : \frac{1}{BF} : \frac{1}{CF}$ and the remaining three are the sidelines of the cevian triangle of Q. These four lines are the directrices of the sought circumconics and their construction is therefore easy to realize. See Figure 5.

This shows that one can find six other circumconics with focus at a Fermat point but, in this case, this focus is not the perspector.

5. Some related loci

We now generalize some of the particularities of the Simmons inconics and present several higher degree curves which all contain the Fermat points.

5.1. *Directrices and trilinear polars.* We have seen that these Simmons inconics are quite remarkable in the sense that the directrix corresponding to the perspector/focus F (which is the polar line of F in the conic) is also the trilinear polar of F. The generalization gives the following

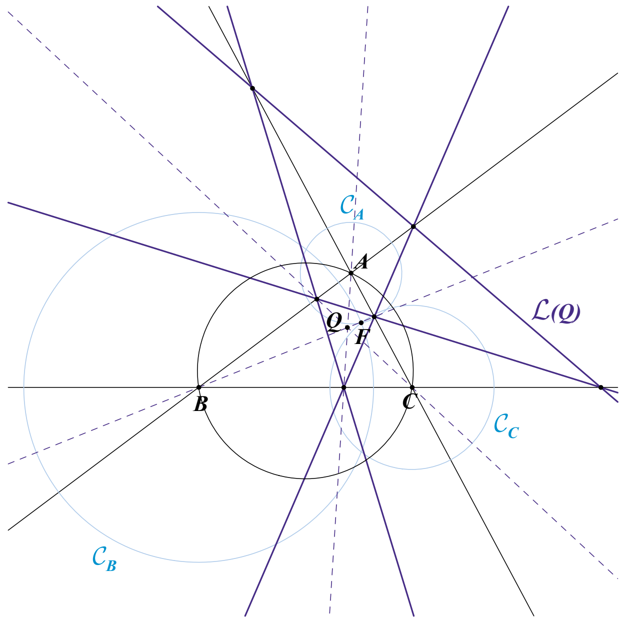

Figure 5. Directrices of circumconics with given focus

Theorem 5. *The locus of the focus F of the inconic such that the corresponding directrix is parallel to the trilinear polar of F is the Euler-Morley quintic Q003.*

Q003 is a very remarkable curve with equation

$$\sum_{\text{cyclic}} a^2(S_B\, y - S_C\, z)y^2z^2 = 0$$

which (at the time this paper is written) contains 70 points of the triangle plane. See [3] and [4].

In Figure 6, we have the inconic with focus F at one of the extraversions of X_{1156} (on the Euler-Morley quintic).

5.2. *Perspector lying on one axis.* The Simmons inconics (or circumconics) have their perspectors at a focus hence on an axis. More generally,

Theorem 6. *The locus of the perspector P of the inconic (or circumconic) such that P lies on one of its axis is the Stothers quintic Q012.*

The Stothers quintic Q012 has equation

$$\sum_{\text{cyclic}} a^2(y - z)(x^2 - yz)yz = 0.$$

Q012 is also the locus of point M such that the circumconic and inconic with same perspector M have parallel axes, or equivalently such that the pencil of conics generated by these two conics contains a circle. See [3].

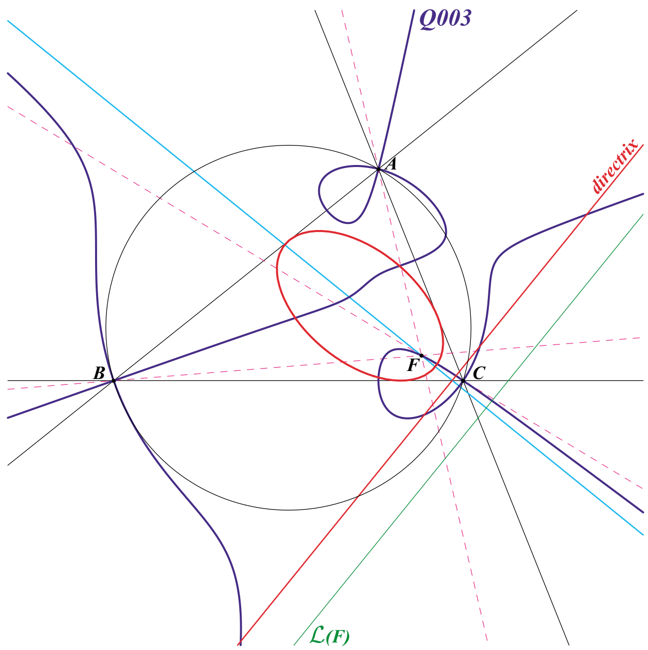

Figure 6. An inconic with directrix parallel to the trilinear polar of the focus

The center of the inconic in Theorem 6 must lie on the complement of the isotomic conjugate of Q012, another quartic with equation

$$\sum_{\text{cyclic}} a^2(y+z-x)(y-z)(y^2+z^2-xy-xz)=0.$$

In Figure 7, we have the inconic with perspector X_{673} (on the Stothers quintic) and center X_{3008}.

The center of the circumconic in Theorem 6 must lie on a septic which is the $G-$Ceva conjugate of Q012.

5.3. *Perspector lying on the focal axis.* The focus F, its isogonal conjugate F^* (the other focus), the center Ω (midpoint of FF^*) and the perspector P (the isotomic conjugate of the anticomplement of Ω) of the inconic may be seen as a special case of collinear points. More generally,

Theorem 7. *The locus of the focus F of the inconic such that F, F^* and P are collinear is the bicircular isogonal sextic* Q039.

Q039 is also the locus of point P whose pedal triangle has a Brocard line passing through P. See [3].

Remark. The locus of P such that the polar lines of P and its isogonal conjugate P^* in one of the Simmons inconics are parallel are the two isogonal pivotal cubics K129a and K129b.

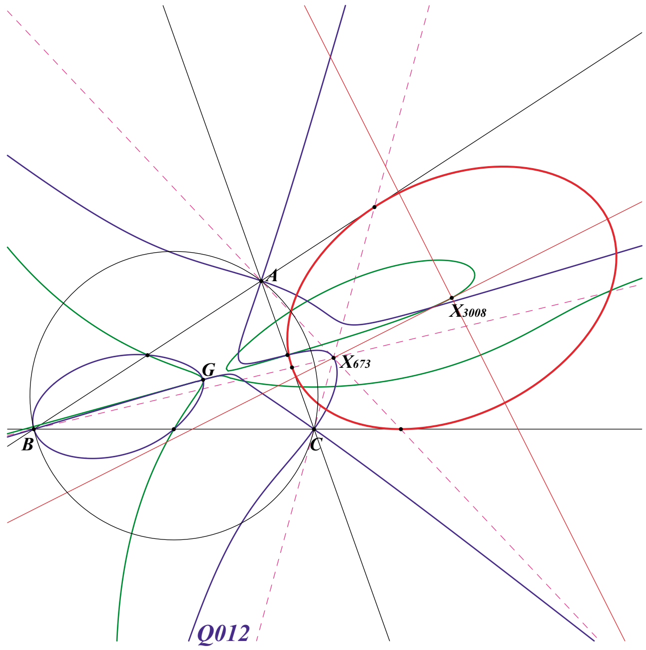

Figure 7. An inconic with perspector on one axis

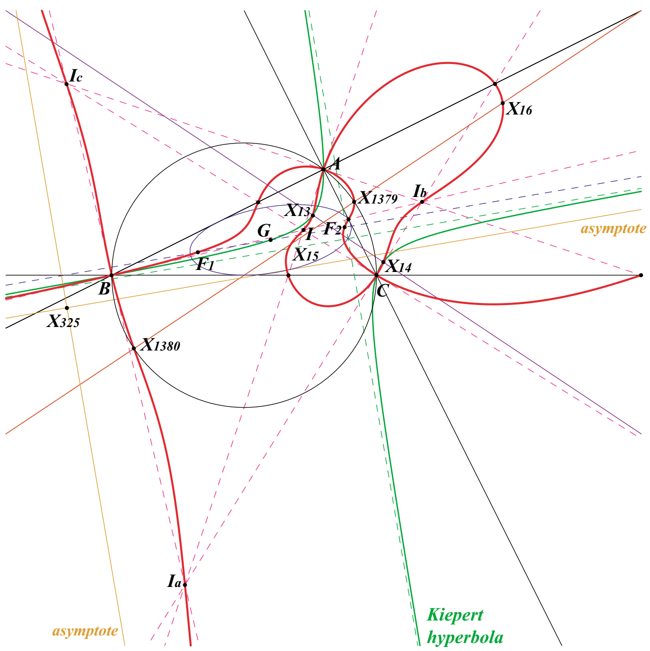

Figure 8. The bicircular isogonal sextic Q039

More precisely, with the conic $\mathcal{S}_{13}$ we obtain K129b = $\mathrm{p}\mathcal{K}(K, X_{396})$ and with the conic $\mathcal{S}_{14}$ we obtain K129a = $\mathrm{p}\mathcal{K}(K, X_{395})$. See [3].

6. Appendices

6.1. In his paper [7], T. C. Simmons has shown that the eccentricity of Σ_{13} is twice that of $\mathcal{S}_{13}$. This is also true for Σ_{14} and $\mathcal{S}_{14}$. The following table gives these eccentricities.

conic	eccentricity
$\mathcal{S}_{13}$	$\dfrac{1}{\sqrt{2(\cot\omega+\sqrt{3})}} \times \dfrac{OH}{\sqrt{\Delta}}$
$\mathcal{S}_{14}$	$\dfrac{1}{\sqrt{2(\cot\omega-\sqrt{3})}} \times \dfrac{OH}{\sqrt{\Delta}}$
Σ_{13}	$\dfrac{2}{\sqrt{2(\cot\omega+\sqrt{3})}} \times \dfrac{OH}{\sqrt{\Delta}}$
Σ_{14}	$\dfrac{2}{\sqrt{2(\cot\omega-\sqrt{3})}} \times \dfrac{OH}{\sqrt{\Delta}}$

where ω is the Brocard angle, Δ the area of ABC and OH the distance between O and H.

6.2. Since Σ_{13} and $\mathcal{S}_{13}$ (or Σ_{14} and $\mathcal{S}_{14}$) have the same focus and the same directrix, it is possible to find infinitely many homologies (perspectivities) transforming these two conics into concentric circles with center X_{13} (or X_{14}) and the radius of the first circle is twice that of the second circle.

The axis of such homology must be parallel to the directrix and its center must be the common focus. Furthermore, the homology must send the directrix to the line at infinity and, for example, must transform the point P_1 (or P_2, see remark 3 at the end of §3) into the infinite point X_{30} of the Euler line or the line $X_{13}X_{15}$.

Let Δ_1 and Δ_2 be the two lines with equations

$$\sum_{\text{cyclic}} (b^2+c^2-2a^2+\sqrt{a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2})\, x = 0$$

and

$$\sum_{\text{cyclic}} (b^2+c^2-2a^2-\sqrt{a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2})\, x = 0.$$

Δ_1 and Δ_2 are the tangents to the Steiner inellipse which are perpendicular to the Euler line. The contacts lie on the line GK and on the circle with center G passing through X_{115}, the center of the Kiepert hyperbola. Δ_1 and Δ_2 meet the Euler line at two points lying on the circle with center G passing through X_{125}, the center of the Jerabek hyperbola.

If we take one of these lines as an axis of homology, the two Simmons circumconics Σ_{13} and Σ_{14} are transformed into two circles Γ_{13} and Γ_{14} having the same radius. Obviously, the two Simmons inconics are also transformed into two circles having the same radius. See Figure 9.

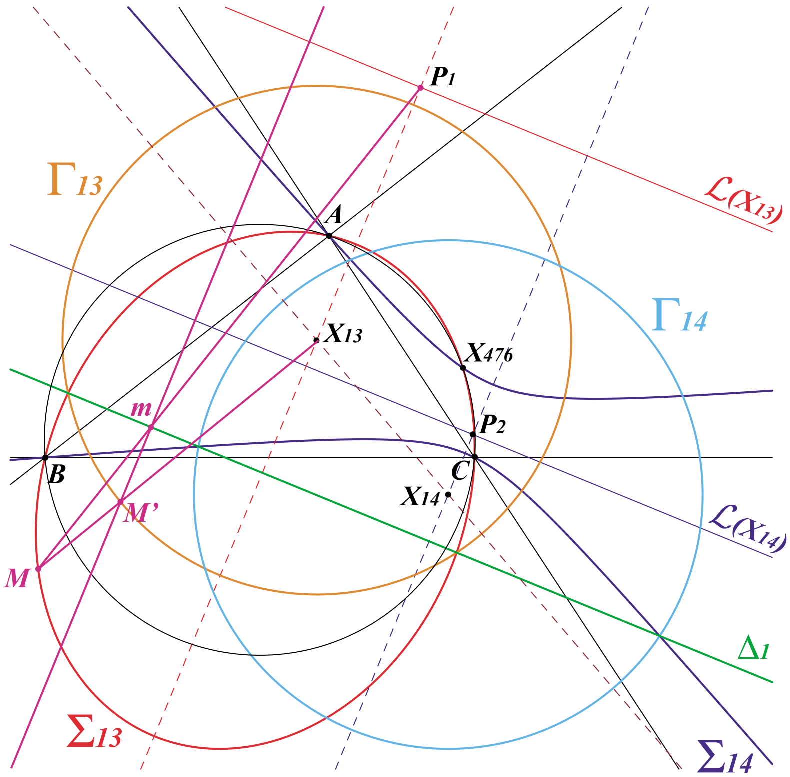

Figure 9. Homologies and circles

For any point M on Σ_{13}, the line MP_1 meets Δ_1 at m. The parallel to the Euler line at m meets the line MX_{13} at M' on Γ_{13}. A similar construction with M on Σ_{14} and P_2 instead of P_1 will give Γ_{14}.

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