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On Some Theorems of Poncelet and Carnot

Huub P.M. van Kempen

Abstract. Some relations in a complete quadrilateral are derived. In connection with these relations some special conics related to the angular points and sides of the quadrilateral are discussed. A theorem of Carnot valid for a triangle is extended to a quadrilateral.

1. Introduction

The scope of Euclidean Geometry was substantially extended during the seventeenth century by the introduction of the discipline of Projective Geometry. Until then geometers were mainly concentrating on the *metric* (or *Euclidean*) properties in which the measure of distances and angles is emphasized. Projective Geometry has no distances, no angles, no circles and no parallelism but concentrates on the *descriptive* (or *projective*) properties. These properties have to do with the relative positional connection of the geometric elements in relation to each other; the properties are unaltered when the geometric figure is subjected to a projection.

Projective Geometry was started by the Grecian mathematician Pappus of Alexandria . After more than thirteen centuries it was continued by two Frenchmen, Desargues and his famous pupil Pascal. The latter one published in 1640 his well-known *Essay pour les coniques*. This short study contains the well-known *hexagrammum mysticum*, nowadays known as Pascal's Theorem. Meanwhile, the related subject of perspective had been studied by architects and artists (Leonardo da Vinci). The further development of Projective Geometry was about two hundred years later, mainly by a French group of mathematicians (Poncelet, Chasles, Carnot, Brianchon and others).

An important tool in Projective Geometry is a semi-algebraic instrument, called the *cross ratio*. This topic was introduced, independently of each other, by Möbius (1827) and Chasles (1829).

In this article we present an (almost forgotten) result of Poncelet [4], obtained in an alternative way and we derive some associated relations (Theorem 1). Furthermore, we extend a theorem by Carnot [1] from a triangle to a complete quadrilateral (Theorem 3).

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We take as starting-point the theorems of Ceva and Pappus-Pascal. The first one is a close companion of the theorem of the Grecian mathematician Menelaus. In the analysis we will follow as much as possible the purist/synthetic approach. It will be shown that this approach leads to surprising results derived along unexpected lines.

2. Proof and extension of a Theorem by Poncelet

Theorem 1. *Let the diagonal points of a complete quadrilateral ABCD be P, Q and R. Let the intersections of PQ with AD and BC be H and F respectively and those of PR with CD and AB be G and E respectively* (*Figure 1*)*. Then*

$$\frac{AE}{EB}\cdot\frac{BF}{FC}\cdot\frac{CG}{GD}\cdot\frac{DH}{HA}=1, \tag{1}$$

$$\frac{AP}{PC}\cdot\frac{CG}{GD}\cdot\frac{DP}{PB}\cdot\frac{BE}{EA}=1, \tag{2}$$

$$\frac{BP}{PD}\cdot\frac{DH}{HA}\cdot\frac{AP}{PC}\cdot\frac{CF}{FB}=1. \tag{3}$$

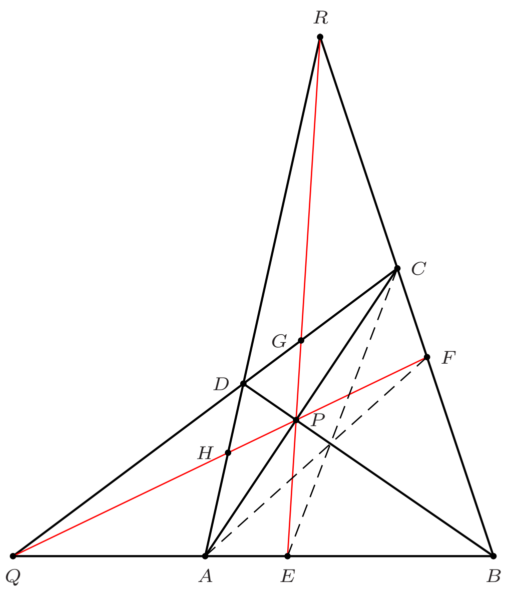

Figure 1

Proof. We apply the Pappus-Pascal theorem to the triples (Q, A, E) and (R, C, F) and find that in triangle ABC the lines AF, BP and CE are concurrent so that by Ceva's theorem

$$\frac{AE}{EB}\cdot\frac{BF}{FC}\cdot\frac{CP}{PA}=1. \tag{4}$$

Similarly with the triples (Q, G, C) and (R, H, A) we find

$$\frac{CG}{GD}\cdot\frac{DH}{HA}\cdot\frac{AP}{PC}=1. \tag{5}$$

Relation (1) immediately follows from (4) and (5). Again in the same way with triples (E, B, Q) and (H, D, R) we find that

$$\frac{AE}{EB} \cdot \frac{BP}{PD} \cdot \frac{DH}{HA} = 1. \tag{6}$$

(2) follows from (5) and (6), and (3) follows from (4) and (6). □

Poncelet [4] has derived relation (1) by using cross ratios.

We now consider a special case of Theorem 1, taking a convex quadrilateral $ABCD$ in which $AB + CD = BC + DA$, so that it is circumscriptable (Figure 2). Let E', F', G' and H' be the points of tangency of the incircle with AB, BC, CD and DA respectively. Clearly a relation similar to (1) holds:

$$\frac{AE'}{E'B} \cdot \frac{BF'}{F'C} \cdot \frac{CG'}{G'D} \cdot \frac{DH'}{H'A} = 1. \tag{7}$$

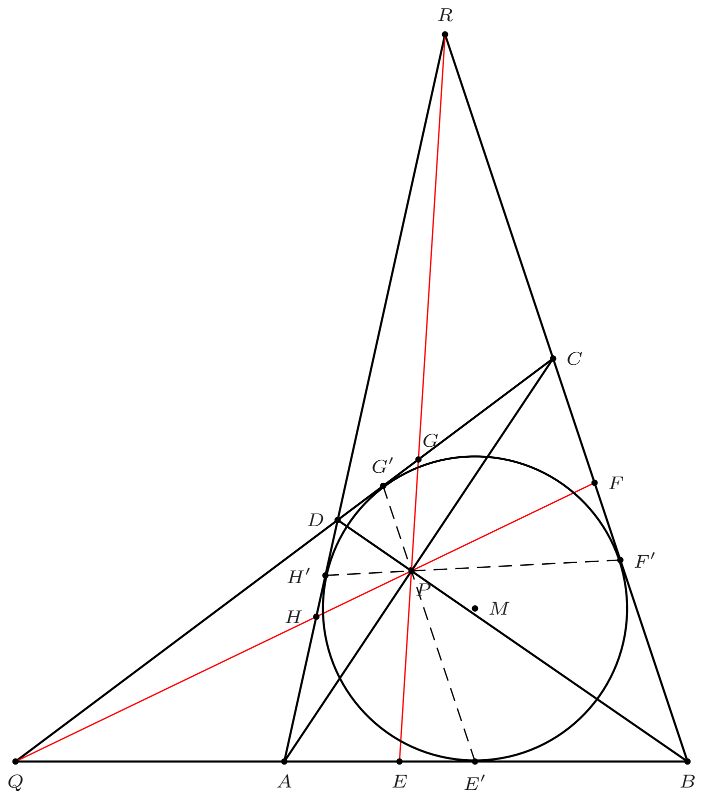

Figure 2

It is well known [5] that the point of intersection of $E'G'$ and $F'H'$ is P. This can be seen for a general quadrilateral with an inscribed conic from subsequent application of Brianchon's theorem to hexagons $AE'BCG'D$ and $BF'CDH'A$. See for instance [2, p.49]. This raises the questions whether or not a relation similar to (1) will hold. We will examine this problem by using Ceva's theorem.

3. Further Analysis

We start with a given quadrilateral $ABCD$ where E and G are arbitrary points on the lines AB and CD respectively. We then construct points F_1 and H_1 on BC and AD respectively such that (1) holds. We can do so by the following construction (Figure 3).

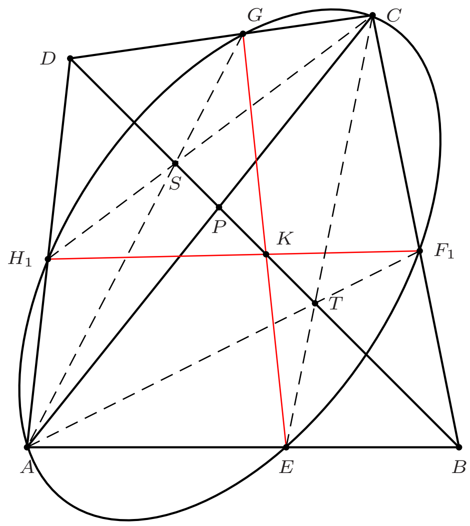

Figure 3

First we consider the triangles ABC and ADC. Let $T = CE \cap BD$ and $F_1 = BC \cap AT$. By Ceva's theorem we have in triangle ABC

$$\frac{AE}{EB} \cdot \frac{BF_1}{F_1C} \cdot \frac{CP}{PA} = 1. \tag{8}$$

Now if $S = AG \cap DB$ and $H_1 = AD \cap CS$, then Ceva's theorem applied to triangle ADC gives

$$\frac{CG}{GD} \cdot \frac{DH_1}{H_1A} \cdot \frac{AP}{PC} = 1. \tag{9}$$

By multiplication of (8) and (9) we find the desired equivalence of (1).

Theorem 2. *If in the quadrilateral $ABCD$ the points E, F_1, G and H_1 lie on AB, BC, CD and DA respectively such that $S = AG \cap CH_1$ and $T = AF_1 \cap CE$ lie on BD, then the points A, E, F_1, C, G and H_1 lie on a conic and $K = EG \cap F_1H_1$ lies on BD.*

Proof. Here we have to switch to the field of Projective Geometry. We will use the cross ratio of pencils in relation to the cross-ratio of ranges. These concepts are extensively described by Eves [3]. Now consider the two pencils (AH_1, AG, AF_1, AE) and (CH_1, CG, CF_1, CE) in Figure 3. We have the cross-ratio equality between ranges and pencils:

$$A(H_1, G; F_1, E) = (D, S; T, B) = (S, D; B, T) = C(H_1, G; F_1, E). \qquad (10)$$

From this equality we see that A, E, F_1, C, G and H_1 lie on a conic. Applying Pascal's theorem to the hexagon AF_1H_1CEG we find that the diagonal BD is the Pascal line and consequently the points S, K and T are collinear. □

By using triangle ABD and triangle CBD instead of triangle ABC and triangle ADC as above, we can also construct F_2 and H_2 such that relation (1) holds (Figure 4). Now we apply Theorem 2, finding that B, F_2, G, D, H_2 and E lie on a conic. Using Pascal's theorem for the hexagon $BGEDF_2H_2$ we find that $L = EG \cap F_2H_2$ lies on AC.

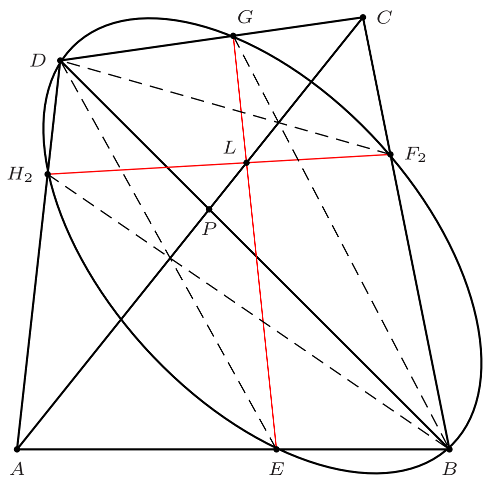

Figure 4

With the help of Theorem 2 we prove an extension of Carnot's theorem in [1] for a triangle to a complete quadrilateral.

Theorem 3. *If in the quadrilateral $ABCD$ the points E, F, G and H lie on AB, BC, CD and DA respectively and EG and FH concur in $P = AC \cap BD$, then (1) is satisfied if and only if there is a conic inscribed in quadrilateral $ABCD$, which touches its sides in the points E, F, G and H.*

Proof. Assume that relation (1) holds. By Theorem 2 we know that $BFGDHE$ and $AEFCGH$ lie on two conics. Let $V = DE \cap BH$ and $W = DF \cap BG$. First we apply Desargues' theorem to triangle GFC and triangle EHV (Figure 5).

The lines GE, FH and CV concur in P. This means that the intersection points of the corresponding sides are collinear. So $U = GF \cap EH$, $B = FC \cap HV$ and $D = GC \cap EV$ are collinear. Next, consider the unique conic Γ through E, F, G and H which is tangent to CD at G. We examine the direction of the tangent to Γ at the point H. Therefore we consider the hexagon $GGEHHF$. We find that

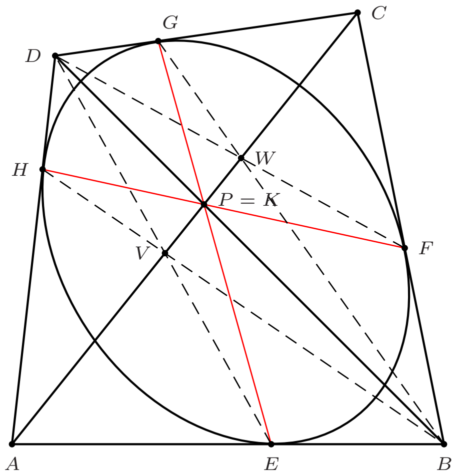

Figure 5

$GE \cap HF = P$ and $GF \cap EH = U$. Since both P and U are collinear with B and D, the line BD is the Pascal line. This means that the tangents to Γ at G and H intersect on BD, which implies that AD is the tangent to Γ at H.

In the same way we prove that the lines AB and BC are tangent to Γ at E and F respectively, which proves the sufficiency part.

Now assume that a conic is tangent to the sides of quadrilateral $ABCD$ at the points E, F, G and H. Note that of course EG and FH intersect in P, as stated earlier. With fixed E, F and G there is exactly one point H^* on AD such that the equivalent version of relation (1) holds. By the sufficiency part this leads to a conic tangent to the sides at E, F, G, and H^*. As these two conics have three double points in common, they must be the same conic. This leads to the conclusion that H and H^* are in fact the same point. This proves the necessity part. $\square$

Applying Theorem 3 to the results of Theorem 2 we find

Corollary 4. *If in the quadrilateral $ABCD$ of Theorem 2 the lines EG and F_1H_1 concur in P, where $P = AC \cap BD$, then F_1H_1 of Figure 3 and F_2H_2 of Figure 4 coincide.*

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Huub P.M. van Kempen: Prins Mauritsplein 17, 2582 NC Den Haag, The Netherlands
E-mail address: hvkempen@versatel.nl