Forum Geometricorum
Volume 6 (2006) 235–240.

FORUM GEOM
ISSN 1534-1178

The Droz-Farny Theorem and Related Topics

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Abstract. At each point P of the Euclidean plane Π, not on the sidelines of a triangle $A_1A_2A_3$ of Π, there exists an involution in the pencil of lines through P, such that each pair of conjugate lines intersect the sides of $A_1A_2A_3$ in segments with collinear midpoints. If $P = H$, the orthocenter of $A_1A_2A_3$, this involution becomes the orthogonal involution (where orthogonal lines correspond) and we find the well-known Droz-Farny Theorem, which says that any two orthogonal lines through H intersect the sides of the triangle in segments with collinear midpoints. In this paper we investigate two closely related loci that have a strong connection with the Droz-Farny Theorem. Among examples of these loci we find the circumcirle of the anticomplementary triangle and the Steiner ellipse of that triangle.

1. The Droz-Farny Theorem

Many proofs can be found for the original Droz-Farny Theorem (for some recent proofs, see [1], [3]). The proof given in [3] (and [5]) probably is one of the shortest: Consider, in the Euclidean plane Π, the pencil $\mathcal{B}$ of parabola's with tangent lines the sides $a_1 = A_2A_3, a_2 = A_3A_1, a_3 = A_1A_2$ of $A_1A_2A_3$, and the line l at infinity. Let P be any point of Π, not on a sideline of $A_1A_2A_3$, and not on l, and consider the tangent lines r and r' through P to a non-degenerate parabola $\mathcal{P}$ of this pencil $\mathcal{B}$. A variable tangent line of $\mathcal{P}$ intersects r and r' in corresponding points of a projectivity (an affinity, i.e. the points at infinity of r and r' correspond), and from this it follows that the line connecting the midpoints of the segments determined by r and r' on a_1 and a_2, is a tangent line of $\mathcal{P}$, through the midpoint of the segment determined on a_3 by r and r'. Next, by the Sturm-Desargues Theorem, the tangent lines through P to a variable parabola of the pencil $\mathcal{B}$ are conjugate lines in an involution $\mathcal{I}$ of the pencil of lines through P, and this involution $\mathcal{I}$ contains in general just one orthogonal conjugate pair. In the following we call these orthogonal lines through P, the orthogonal Droz-Farny lines through P.

Remark that (PA_i, line through P parallel to a_i), $i = 1, 2, 3$ are the tangent lines through P of the degenerate parabola's of the pencil $\mathcal{B}$, and thus are conjugate pairs in the involution $\mathcal{I}$. From this it follows that in the case where $P = H$, the orthocenter of $A_1A_2A_3$, this involution becomes the orthogonal involution in the pencil of lines through H, and we find the Droz-Farny Theorem.

Publication Date: October 2, 2006. Communicating Editor: J. Chris Fisher.

Two other characterizations of the orthogonal Droz-Farny lines through P are obtained as follows: Let X and Y be the points at infinity of the orthogonal Droz-Farny lines through P. Since the two triangles $A_1A_2A_3$ and PXY are circumscribed triangles about a conic (a parabola of the pencil $\mathcal{B}$), their vertices are six points of a conic, namely the rectangular hyperbola through A_1, A_2, A_3, and P (and also through H, since any rectangular hyperbola through A_1, A_2, and A_3, passes through H). It follows that the orthogonal Droz-Farny lines through P are the lines through P which are parallel to the (orthogonal) asymptotes of this rectangular hyperbola through A_1, A_2, A_3, P, and H.

Next, since, if $P = H$, the involution $\mathcal{I}$ is the orthogonal involution, the directrix of any parabola of the pencil $\mathcal{B}$ passes through H, and the orthogonal Droz-Farny lines through any point P are the orthogonal tangent lines through P of the parabola, tangent to a_1, a_2, a_3, and with directrix PH.

2. The first locus

Let us recall some basic properties of trilinear (or normal) coordinates (see for instance [4]). Trilinear coordinates (x_1, x_2, x_3), with respect to a triangle A_1A_2A with side-lenghts l_1, l_2, l_3, of any point P of the Euclidean plane, are homogeneous projective coordinates, in the Euclidean plane, for which the vertices A_1, A_2, A_3 are the basepoints and the incenter I of the triangle the unit point. The line at infinity has in trilinear coordinates the equation $l_1x_1 + l_2x_2 + l_3x_3 = 0$. The centroid G of $A_1A_2A_3$ has trilinear coordinates $(\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_3})$, the orthocenter H is $(\frac{1}{\cos A_1}, \frac{1}{\cos A_2}, \frac{1}{\cos A_3})$, the circumcenter O is $(\cos A_1, \cos A_2, \cos A_3)$, the incenter I is $(1, 1, 1)$, and the Lemoine (or symmedian) point K is (l_1, l_2, l_3).

If X has trilinear coordinates (x_1, x_2, x_3) with respect to $A_1A_2A_3$, and if d_i is the "'signed"' distance from X to the side a_i (i.e. d_i is positive or negative, according as X lies on the same or opposite side of a_i as A_i), then, if F is the area of $A_1A_2A_3$, we have $d_i = \frac{2Fx_i}{l_1x_1+l_2x_2+l_3x_3}$, $i = 1, 2, 3$, and (d_1, d_2, d_3) are the *actual trilinear coordinates* of X with respect to $A_1A_2A_3$. Remark that $l_1d_1 + l_2d_2 + l_3d_3 = 2F$.

Our first locus is defined as follows ([5]):

Consider a fixed point P, not on a sideline of $A_1A_2A_3$, and not at infinity, with actual trilinear coordinates $(\delta_1, \delta_2, \delta_3)$ with respect to $A_1A_2A_3$, and suppose that s is a given real number and the set of points of the plane for which the distances d_i from (x_1, x_2, x_3) to a_i are connected by the equation

$$\frac{l_1}{\delta_1}d_1^2 + \frac{l_2}{\delta_2}d_2^2 + \frac{l_3}{\delta_3}d_3^2 = s. \tag{1}$$

Using $d_i = \frac{2F_ix_i}{l_1x_1+l_2x_2+l_3x_3}$, we see that the set is given by the equation

$$4F^2(\frac{l_1}{\delta_1}x_1^2 + \frac{l_2}{\delta_2}x_2^2 + \frac{l_3}{\delta_3}x_3^2) - s(l_1x_1 + l_2x_2 + l_3x_3)^2 = 0, \tag{2}$$

or, if we use general trilinear coordinates (p_1, p_2, p_3) of P:

$$2F(\frac{l_1}{p_1}x_1^2 + \frac{l_2}{p_2}x_2^2 + \frac{l_3}{p_3}x_3^2)(l_1p_1 + l_2p_2 + l_3p_3) - s(l_1x_1 + l_2x_2 + l_3x_3)^2 = 0. \quad (3)$$

We denote this conic by $\mathcal{K}(P, \Delta, s)$: it is the conic determined by (1) and (2), where $(\delta_1, \delta_2, \delta_3)$ are the actual trilinear coordinates of P with regard to $\Delta = A_1A_2A_3$, and also by (3), where (p_1, p_2, p_3) are any triple of trilinear coordinates of P with regard to Δ, and by the value of s. For P and Δ fixed and s allowed to vary, the conics $\mathcal{K}(\mathcal{P}, \Delta, s)$ belong to a pencil, and a straightforward calculation shows that all conics of this pencil have center P, and have the same points at infinity, which means that they have the same asymptotes and the same axes.

The conics $\mathcal{K}(P, \Delta, s)$ can be (homothetic) ellipses or hyperbola's: this depends on the location of P with regard to Δ, and again a straightforward calculation shows that we find ellipses or hyperbola's, according as the product $\delta_1\delta_2\delta_3 > 0$ or < 0.

Next, the *medial triangle* of $\Delta = A_1A_2A_3$ is the triangle whose vertices are the midpoints of the sides of Δ, and the *anticomplementary triangle* $A_1^{-1}A_2^{-1}A_3^{-1}$ of Δ is the triangle whose medial triangle is Δ. An easy calculation shows that the trilinear coordinates of the vertices A_1^{-1}, A_2^{-1}, and A_3^{-1} of this anticomplementary triangle are $(-l_2l_3, l_3l_1, l_1l_2)$, $(l_2l_3, -l_3l_1, l_1l_2)$, and $(l_2l_3, l_3l_1, -l_1l_2)$, respectively.

Lemma 1. *The locus $\mathcal{K}(P, \Delta, S)$ of the points for which the distances d_1, d_2, d_3 to the sides a_1, a_2, a_3 of $\Delta = A_1A_2A_3$ are connected by*

$$\frac{l_1}{\delta_1}d_1^2 + \frac{l_2}{\delta_2}d_2^2 + \frac{l_3}{\delta_3}d_3^2 = 4F^2(\frac{1}{l_1\delta_1} + \frac{1}{l_2\delta_2} + \frac{1}{l_3\delta_3}) \ = S,$$

where $(\delta_1, \delta_2, \delta_3)$ are the actual trilinear coordinates of a given point P, is the conic with center P, and circumscribed about the anticomplementary triangle $A_1^{-1}A_2^{-1}A_3^{-1}$ of Δ.

Proof. Substituting the coordinates $(-\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_3})$, or $(\frac{1}{l_1}, -\frac{1}{l_2}, \frac{1}{l_3})$, or $(\frac{1}{l_1}, \frac{1}{l_2}, -\frac{1}{l_3})$ of A_1^{-1}, A_2^{-1}, and A_3^{-1}, in (2), we find immediately that

$$s = S = 4F^2(\frac{1}{l_1\delta_1} + \frac{1}{l_2\delta_2} + \frac{1}{l_3\delta_3}).$$

□

3. The second locus

We work again in the Euclidean plane Π, with trilinear coordinates with respect to $\Delta = A_1A_2A_3$. Assume that $P(p_1, p_2, p_3)$ is a point of Π, not at infinity and not on a sideline of Δ. We look for the locus of the points Q of Π, such that the points $Q_i = q_i \cap a_i,\ i = 1, 2, 3$, where q_i is the line through Q, parallel to PA_i, are collinear. This locus was the subject of the paper [2]. Since $l_1x_1 + l_2x_2 + l_3x_3 = 0$ is the equation of the line at infinity, the point at infinity of PA_1 has coordinates $(l_2p_2 + l_3p_3, -l_1p_2, -l_1p_3)$, and if we give Q the coordinates (x_1, x_2, x_3), we find after an easy calculation that Q_1 has coordinates $(0, l_1p_2x_1 +$

$x_2(l_2p_2+l_3p_3), x_1l_1p_3+x_3(l_2p_2+l_3p_3))$. In the same way, we find for the coordinates of Q_2, and of Q_3: $(x_1(l_1p_1+l_3p_3)+p_1x_2l_2, 0, p_3x_2l_2+x_3(l_1p_1+l_3p_3))$, and $(x_1(l_1p_1+l_2p_2)+x_3p_1l_3, x_2(l_1p_1+l_2p_2)+x_3p_2l_3, 0)$, respectively.

Next, after a rather long calculation, and deleting the singular part $l_1x_1+l_2x_2+l_3x_3=0$, the condition that Q_1, Q_2, and Q_3 are collinear, gives us the following equation for the locus of the point Q:

$$p_3(l_1p_1+l_2p_2)x_1x_2+p_1(l_2p_2+l_3p_3)x_2x_3+p_2(l_3p_3+l_1p_1)x_3x_1=0. \quad (4)$$

This is our second locus, and we denote this conic, circumscribed about $A_1A_2A_3$, by $\mathcal{C}(P,\Delta)$, where $\Delta = A_1A_2A_3$, and where P is the point with trilinear coordinates (p_1,p_2,p_3) with regard to Δ.

Lemma 2. *The center M of $\mathcal{C}(P,\Delta)$ has trilinear coordinates*

$$(l_2l_3(l_2p_2+l_3p_3), l_3l_1(l_3p_3+l_1p_1), l_1l_2(l_1p_1+l_2p_2)).$$

It is the image $f(P)$, where f is the homothety with center G, the centroid of Δ, and homothetic ratio $-\frac{1}{2}$, or, in other words: $2\vec{GM} = -\vec{GP}$.

Proof. An easy calculation shows that the polar point of this point M with regard to the conic (4) is indeed the line at infinity, with equation $l_1x_1+l_2x_2+l_3x_3=0$. Moreover, if P_∞ is the point at infinity of the line PG, the equation $2\vec{GM} = -\vec{GP}$ is equivalent with the equality of the cross-ratio $(MPGP_\infty)$ to $-\frac{1}{2}$. Next, choose on the line PG homogeneous projective coordinates with basepoints $P(1,0)$ and $G(0,1)$, and give P_∞ coordinates (t_1,t_2) (thus $P_\infty = t_1P+t_2G$), then $(t_1,t_2) = (-3, l_1p_1+l_2p_2+l_3p_3)$ and the projective coordinates (t_1', t_2') of M follow from

$$(MPGP_\infty) = \frac{\begin{vmatrix} t_1' & t_2' \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}} : \frac{\begin{vmatrix} t_1' & t_2' \\ -3 & l_1p_1+l_2p_2+l_3p_3 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ -3 & l_1p_1+l_2p_2+l_3p_3 \end{vmatrix}} = -\frac{1}{2},$$

which gives $(t_1', t_2') = (-1, l_1p_1+l_2p_2+l_3p_3)$. $\square$

Remark that the second part of the proof also follows from the connection between trilinears (x_1,x_2,x_3) for a point with respect to $A_1A_2A_3$ and trilinears (x_1', x_2', x_3') for the same point with respect to the medial triangle of $A_1A_2A_3$ (see [4, p.207]):

$$\begin{cases} x_1 = l_2l_3(l_2x_2'+l_3x_3') \\ x_2 = l_3l_1(l_3x_3'+l_1x_1') \\ x_3 = l_1l_2(l_1x_1'+l_2x_2'). \end{cases}$$

4. The connection between the Droz-Farny -lines and the conics

Recall from §2 that

$$S = 4F^2\left(\frac{1}{l_1\delta_1}+\frac{1}{l_2\delta_2}+\frac{1}{l_3\delta_3}\right) = 2F(l_1p_1+l_2p_2+l_3p_3)\left(\frac{1}{l_1p_1}+\frac{1}{l_2p_2}+\frac{1}{l_3p_3}\right),$$

where F is the area of $\Delta = A_1A_2A_3$, (p_1, p_2, p_3) are trilinear coordinates of P with regard to $\Delta A_1A_2A_3$, and where $(\delta_1, \delta_2, \delta_3)$ are the actual trilinear coordinates of P with respect to this triangle.
Furthermore, in the foregoing section, f is the homothety with center G and homothetic ratio $-\frac{1}{2}$. Remark that $f^{-1}(\Delta)$ is the anticomplementary triangle Δ^{-1} of Δ. We have:

Theorem 3. (1) *The conics $\mathcal{K}(P, \Delta, S)$ and $\mathcal{C}(f^{-1}(P), f^{-1}(\Delta))$ coincide.*
(2) *The common axes of the conics $\mathcal{K}(P, \Delta, s)$, $s \in R$, and of the conic $\mathcal{C}(f^{-1}(P), f^{-1}(\Delta))$ are the orthogonal Droz-Farny -lines through P, with regard to $\Delta = A_1A_2A_3$.*

Proof. (1) Because of Lemma 1 and 2, both conics have center P and are circumscribed about the complementary triangle $f^{-1}(\Delta)$ of $A_1A_2A_3$.

(2) For the conic with center P, circumscribed about the anticomplementary triangle of $A_1A_2A_3$, it is clear that (PA_i, line through P, parallel to a_i), $i = 1, 2, 3$, are conjugate diameters. And the result follows from section 1. □

5. Examples

5.1. If $P = H$, the orthocenter of $\Delta = A_1A_2A_3$, which is also the circumcenter of its anticomplementary triangle Δ^{-1}, the conics $\mathcal{K}(H, \Delta, s)$ are circles with center H, since any two orthogonal lines through H are axes of these conics. In particular, $\mathcal{K}(H, \Delta, S)$, where $S = 2F(\frac{\cos A_1}{l_1} + \frac{\cos A_2}{l_2} + \frac{\cos A_3}{l_3})(\frac{l_1}{\cos A_1} + \frac{l_2}{\cos A_2} + \frac{l_3}{\cos A_3})$, is the circumcircle of Δ^{-1} and it is the locus of the points for which the distances d_1, d_2, d_3 to the sides of Δ are related by

$$\begin{aligned}&(l_1 \cos A_1)d_1^2 + (l_2 \cos A_2)d_2^2 + (l_3 \cos A_3)d_3^2\\ =&4F^2(\frac{\cos A_1}{l_1} + \frac{\cos A_2}{l_2} + \frac{\cos A_3}{l_3})\\ =&2F^2\frac{l_1^2 + l_2^2 + l_3^2}{l_1l_2l_3},\end{aligned}$$

or equivalently,

$$l_1^2(l_2^2 + l_3^2 - l_1^2)d_1^2 + l_2^2(l_3^2 + l_1^2 - l_2^2)d_2^2 + l_3^2(l_1^2 + l_2^2 - l_3^2)d_3^2 = 4F^2(l_1^2 + l_2^2 + l_3^2).$$

Moreover, $\mathcal{C}(f^{-1}(H), \Delta^{-1})$ is also the circumcirle of Δ^{-1}, which is easily seen from the fact that this circumcircle is the locus of the points for which the feet of the perpendiculars to the sides of Δ^{-1} are collinear. Remark that $f^{-1}(H)$, the orthocenter of Δ^{-1}, is the de Longchamps point $X(20)$ of Δ.

5.2. If $P = K(l_1, l_2, l_3)$, the Lemoine point of $\Delta = A_1A_2A_3$, then $\mathcal{K}(K, \Delta, S)$, with

$$\begin{aligned}S =&2F(\frac{1}{l_1^2} + \frac{1}{l_2^2} + \frac{1}{l_3^2})(l_1^2 + l_2^2 + l_3^2)\\ =&2F^2\frac{(l_2^2l_3^2 + l_3^2l_1^2 + l_1^2l_2^2)(l_1^2 + l_2^2 + l_3^2)}{l_1^2l_2^2l_3^2},\end{aligned}$$

is the locus of the points for which the distances d_1, d_2, d_3 to the sides of Δ are related by $d_1^2 + d_2^2 + d_3^2 = 4F^2(\frac{1}{l_1^2} + \frac{1}{l_2^2} + \frac{1}{l_3^2})$, and it is the ellipse with center K, circumscribed about Δ^{-1}. Moreover, the locus $\mathcal{C}(f^{-1}(K), \Delta^{-1})$, where $f^{-1}(K)$ is the Lemoine point of Δ^{-1} (or $X(69)$ with coordinates

$$(l_2l_3(l_2^2 + l_3^2 - l_1^2), l_3l_1(l_3^2 + l_1^2 - l_2^2), l_1l_2(l_1^2 + l_2^2 - l_3^2)),$$

is the same ellipse. The axes of this ellipse are the orthogonal Droz-Farny lines through K with respect to Δ.

5.3. If $P = G(\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_3})$, the centroid of $\Delta = A_1A_2A_3$, then $\mathcal{K}(G, \Delta, S)$, with $S = 18F$, is the locus of the points for which the distances d_1, d_2, d_3 to the sides of Δ are related by $l_1^2d_1^2 + l_2^2d_2^2 + l_3^2d_3^2 = 12F^2$, and it is the ellipse with center G, circumscribed about Δ^{-1}, i.e., it is the Steiner ellipse of Δ^{-1}, since G is also the centroid of Δ^{-1}. The locus $\mathcal{C}(G, \Delta^{-1})$ is also this Steiner ellipse and its axes are the orthogonal Droz-Farny lines through G with respect to Δ.

5.4. If $P = I(1, 1, 1)$, the incenter of $\Delta = A_1A_2A_3$, then $\mathcal{K}(I, \Delta, S)$, with $S = 2F(l_1+l_2+l_3)(\frac{1}{l_1}+\frac{1}{l_2}+\frac{1}{l_3})$, is the locus of the points for which the distances d_1, d_2, d_3 to the sides of Δ are related by $l_1d_1^2 + l_2d_2^2 + l_3d_3^2 = 4F^2(\frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3})$, and it is the ellipse with center I, circumscribed about Δ^{-1}. Moreover the locus $\mathcal{C}(f^{-1}(I), \Delta^{-1})$, where $f^{-1}(I)$ is the incenter of Δ^{-1} (which is center $X(8)$ of Δ, the Nagel point with coordinates $(\frac{l_2+l_3-l_1}{l_1}, \frac{l_3+l_1-l_2}{l_2}, \frac{l_1+l_2-l_3}{l_3}))$ is the same ellipse. The axes of this ellipse are the orthogonal Droz-Farny lines through I with respect to Δ.

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