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On Butterflies Inscribed in a Quadrilateral

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Abstract. We explore a configuration consisting of two quadrilaterals which share the intersection of diagonals. We prove results analogous to the Sidney Kung's Butterfly Theorem for Quadrilaterals in [2].

1. The extended butterfly theorem for quadrilaterals

In this note we consider some properties of pairs of quadrilaterals $ABCD$ and $A'B'C'D'$ with A', B', C' and D' on lines AB, BC, CD, and DA respectively.

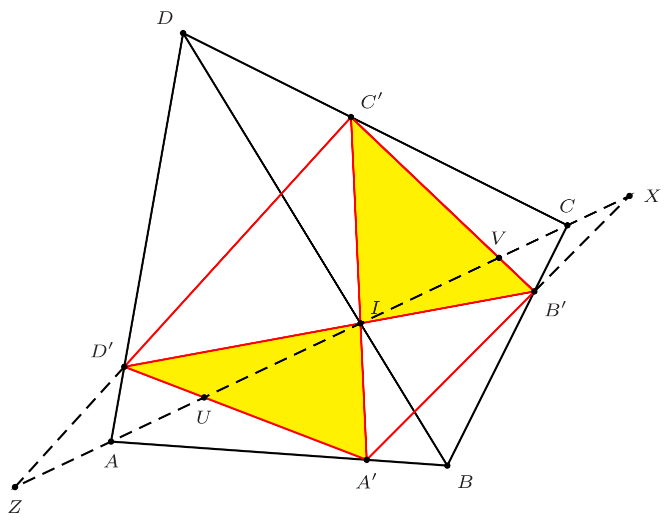

Figure 1

The segments AC, $A'C'$ and $B'D'$ are analogous to the three chords from the classical butterfly theorem (see [1] for an extensive overview of its many proofs and generalizations).

When the intersection of the lines $A'C'$ and $B'D'$ is the intersection I of the lines AC and BD, *i.e.*, when $ABCD$ and $A'B'C'D'$ share the same intersection of diagonals, in [2] the following equality, known as the Butterfly Theorem for Quadrilaterals, was established:

$$\frac{|AU|}{|UI|} \cdot \frac{|IV|}{|VC|} = \frac{|AI|}{|IC|}, \tag{1}$$

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where U and V are intersections of the line AC with the lines $D'A'$ and $B'C'$ (see Figure 1).

For the intersections $X = AC \cap A'B'$ and $Z = AC \cap C'D'$, in this situation, we have similar relations

$$\frac{|XA|}{|AI|} \cdot \frac{|IC|}{|CZ|} = \frac{|XI|}{|IZ|},$$

and

$$\frac{|XU|}{|UI|} \cdot \frac{|IV|}{|VZ|} = \frac{|XI|}{|IZ|}.$$

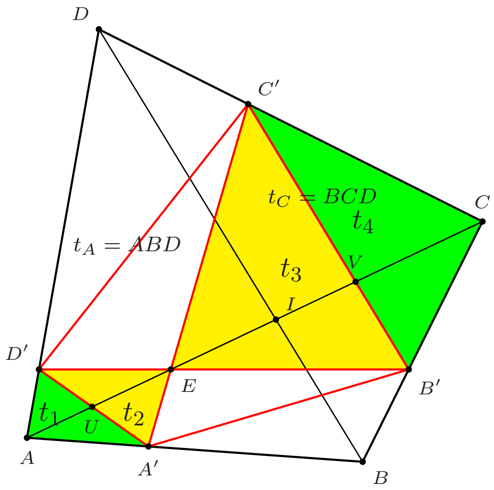

Figure 2

Our first result is the observation that (1) holds when the diagonals of $A'B'C'D'$ intersect at a point E on the diagonal AC of $ABCD$, not necessarily the intersection $I = AC \cap BD$ (see Figure 2).

Theorem 1. *Let $A'B'C'D'$ be an inscribed quadrilateral of $ABCD$, and $E = A'C' \cap B'C'$, $I = AC \cap BD$, $U = AC \cap D'A'$, $V = AC \cap B'C'$. If E lies on the line AC, then*

$$\frac{|AU|}{|UE|} \cdot \frac{|EV|}{|VC|} = \frac{|AI|}{|IC|}. \tag{2}$$

Proof. We shall use analytic geometry of the plane. Without loss of generality we can assume that $A(0,0)$, $B(f,g)$, $C(1,0)$ and $D(p,q)$ for some real numbers f, g, p and q. The points $A'\left(\frac{fu}{u+1}, \frac{gu}{u+1}\right)$ and $B'\left(\frac{f+v}{v+1}, \frac{g}{v+1}\right)$ divide segments AB and BC in ratios u and v, which are real numbers different from -1. Let the rectangular coordinates of the point E be $(h,\ k)$. Then the vertices C' and D' are intersections of the lines CD and DA with the lines $A'E$ and $B'E$, respectively. Their coordinates are a bit more complicated. Next, we determine the points U and

V as intersections of the line AC with the lines $A'D'$ and $B'C'$ and with a small help from Maple V find that the difference

$$\begin{aligned}\mathcal{D} :=& \frac{|AU|^2}{|UE|^2} \cdot \frac{|EV|^2}{|VC|^2} - \frac{|AI|^2}{|IC|^2} \\ =& \frac{(fq - gp)^2 \cdot k \cdot P_5(h,k)}{Q_4(h,k) \cdot (g + fq - q - gp)^2 \cdot (guh + (u + 1 - fu)k - gu)^2},\end{aligned}$$

where $P_5(h,k)$ and $Q_4(h,k)$ are polynomials of degrees 5 and 4 respectively in variables h and k. Both are somewhat impractical to write down explicitly. However, since k is a factor in the numerator we see that when $k = 0$, *i.e.*, when the point E is on the line AC, the difference $\mathcal{D}$ is zero so that the extended Butterfly Theorem for Quadrilaterals holds. □

Remark. There is a version of the above theorem where the points U and V are intersections $U = AE \cap A'D'$ and $V = CE \cap B'C'$. The difference $\mathcal{D}$ in this case is the quotient

$$\frac{-(fq - gp)^2 \cdot k \cdot P_1(h,k) \cdot P_2(h,k)}{(qh - pk)^2 \cdot (g + fq - q - gp)^2 \cdot (guh + (u + 1 - fu)k - gu)^2},$$

where

$$\begin{aligned}P_1(h,k) =& (u(q - g) + q(1 + uv))h + (uv(1 - p) + u(f - p) - p)k \\ & - u(qv - gp + fq), \\ P_2(h,k) =& 2qguh^2 + (gu - 2ugp - 2fqu + uq - vuq + q)\,hk - \\ & (p + uv + up + uf - pvu - 2puf)\,k^2 - 2guqh + u\,(qv + gp + fq)\,k.\end{aligned}$$

Note that these are linear and quadratic polynomials in h and k. In other words, the extended Butterfly Theorem for Quadrilaterals holds not only for points E on the line AC but also when the point E is on a line through D (with the equation $P_1(h,k) = 0$) and on a conic through A, C and D (with the equation $P_2(h,k) = 0$).

Moreover, in this case we can easily prove the following converse of Theorem 1.

If the relation (2) holds when the points A' and B' divide segments AB and BC both in the ratio $1:3$, $1:1$ or $3:1$, then the point E lies on the line AC.

Indeed, if we substitute for $u = v = \frac{1}{3}$, 1 or 3 both into P_1 and P_2 we get three equations whose only common solutions in h and k are coordinates of points A, C, and D (which are definitely excluded as possible solutions).

2. A relation involving areas

Let us introduce shorter notation for six triangles in this configuration: $t_1 = D'AA'$, $t_2 = D'A'E$, $t_3 = EB'C'$, $t_4 = CC'B'$, $t_A = ABD$ and $t_C = CDB$ (see Figure 2).

Our next result shows that the above relationship also holds for areas of these triangles.

Theorem 2. (a) *If E lies on the line AC, then*

$$\frac{\mathtt{area}(t_1)}{\mathtt{area}(t_2)} \cdot \frac{\mathtt{area}(t_3)}{\mathtt{area}(t_4)} = \frac{\mathtt{area}(t_A)}{\mathtt{area}(t_C)}. \tag{3}$$

(b) *If the relation (3) holds when the points A' and B' divide AB and BC both in the ratio $1:2$ or $2:1$, then the point E lies on the line AC.*

Proof. If we keep the same assumptions and notation from the proof of Theorem 1, then

$$\frac{\mathtt{area}(t_1)}{\mathtt{area}(t_2)} \cdot \frac{\mathtt{area}(t_3)}{\mathtt{area}(t_4)} - \frac{\mathtt{area}(t_A)}{\mathtt{area}(t_C)}$$
$$= \frac{(fq-gp)\cdot k \cdot P_1(h,k)}{(qh-pk)\cdot(g+fq-q-gp)\cdot(guh+(u+1-fu)k-gu)}.$$

Hence, (a) is clearly true because $k=0$ when the point E is on the line AC.

On the other hand, for (b), when $u=\frac{1}{2}$ and $v=\frac{1}{2}$ then

$$\mathcal{E}_1 = 4\cdot P_1(h,k) = (2g-7q)\,h + (7p-2f-1)\,k + q - 2gp + 2fq,$$

while for $u=2$ and $v=2$ then

$$\mathcal{E}_2 = P_1(h,k) = (2g-7q)\,h + (7p-2f-4)\,k + 2fq + 4q - 2gp.$$

The only solution of the system

$$\mathcal{E}_1 = 0, \qquad \mathcal{E}_2 = 0$$

is $(h,h)=(p,q)$, *i.e.*, $E=D$. However, for this solution the triangle t_2 degenerates to a segment so that its area is zero which is unacceptable. □

3. Other relations

Note that the above theorem holds also for (lengths of) the altitudes $h(A,t_1)$, $h(E,t_2)$, $h(E,t_3)$, $h(C,t_4)$, $h(A,t_A)$ and $h(C,t_C)$ because (for example) $\mathtt{area}(t_1)=\frac{1}{2}h(A,t_1)\cdot|D'A'|$.

Let $G(t)$ denote the centroid of the triangle $t=ABC$, and $\varepsilon(A,t)$ the distance of $G(t)$ from the side BC opposite to the vertex A. Since $\varepsilon(A,t)=\frac{2\mathtt{area}(t)}{3|BC|}$ there is a version of the above theorem for the distances $\varepsilon(A,t_1)$, $\varepsilon(E,t_2)$, $\varepsilon(E,t_3)$, $\varepsilon(C,t_4)$, $\varepsilon(A,t_A)$ and $\varepsilon(C,t_C)$.

For a triangle t let $R(t)$ denote the radius of its circumcircle. The following theorem shows that the radii of circumcircles of the six triangles satisfy the same pattern without any restrictions on the point E.

Theorem 3. $\frac{R(t_1)}{R(t_2)}\cdot\frac{R(t_3)}{R(t_4)} = \frac{R(t_A)}{R(t_C)}$.

Proof. Let us keep again the same assumptions and notation from the proof of Theorem 1. Since $R(t)=\frac{\text{product of side lengths}}{4\mathtt{area}(t)}$, we see that the square of the circumradius of a triangle with vertices in the points (x,a), (y,b) and (z,c) is given by

$$\frac{[(y-z)^2+(b-c)^2]\cdot[(z-x)^2+(c-a)^2]\cdot[(x-y)^2+(a-b)^2]}{4(x(b-c)+y(c-a)+z(a-b))^2}.$$

Applying this formula we find $R(t_1)^2$, $R(t_2)^2$, $R(t_3)^2$, $R(t_4)^2$, $R(t_A)^2$ and $R(t_C)^2$. In a few seconds Maple V verifies that the difference $\frac{R(t_1)^2}{R(t_2)^2} \cdot \frac{R(t_3)^2}{R(t_4)^2} - \frac{R(t_A)^2}{R(t_C)^2}$ is equal to zero. $\square$

For a triangle t let $H(t)$ denote its orthocenter. In the next result we look at the distances of a particular vertex of the six triangles from its orthocenter. Again the pattern is independent from the position of the point E.

Theorem 4. $\frac{|AH(t_1)|}{|EH(t_2)|} \cdot \frac{|EH(t_3)|}{|CH(t_4)|} = \frac{|AH(t_A)|}{|CH(t_C)|}$.

Proof. This time one can see that $|AH(t)|^2$ for the triangle $t = ABC$ with the vertices in the points (x, a), (y, b) and (z, c) is given as

$$\frac{[(y-z)^2 + (b-c)^2] \cdot [x^2 + a^2 + yz - x(y+z) + bc - a(b+c)]^2}{(x(b-c) + y(c-a) + z(a-b))^2}.$$

Applying this formula we find $|AH(t_1)|^2$, $|EH(t_2)|^2$, $|EH(t_3)|^2$, $|CH(t_4)|^2$, $|AH(t_A)|^2$, and $|CH(t_C)|^2$. In a few seconds Maple V verifies that the difference $\frac{|AH(t_1)|^2}{|EH(t_2)|^2} \cdot \frac{|EH(t_3)|^2}{|CH(t_4)|^2} - \frac{|AH(t_A)|^2}{|CH(t_C)|^2}$ is equal to zero. $\square$

Let $O(t)$ be the circumcenter of the triangle $t = ABC$. Let $\delta(A, t)$ denote the distance of $O(t)$ from the side BC opposite to the vertex A. Since $|AH(t)| = 2\delta(A, t)$ there is a version of the above theorem for the distances $\delta(A, t_1)$, $\delta(E, t_2)$, $\delta(E, t_3)$, $\delta(C, t_4)$, $\delta(A, t_A)$ and $\delta(C, t_C)$.

References

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