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On Triangles with Vertices on the Angle Bisectors

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Abstract. We study interesting properties of triangles whose vertices are on the three angle bisectors of a given triangle. We show that such a triangle is perspective with the medial triangle if and only if it is perspective with the intouch triangle. We present several interesting examples with new triangle centers.

1. Introduction

Let ABC be a given triangle with incenter I. By an I-triangle we mean a triangle UVW whose vertices U, V, W are on the angle bisectors AI, BI, CI respectively. Such triangles are clearly perspective with ABC at the incenter I.

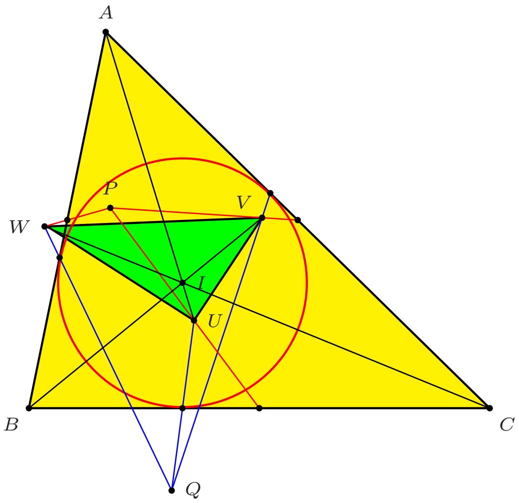

Figure 1.

Theorem 1. *An I-triangle is perspective with the medial triangle if and only if it is perspective with the intouch triangle.*

Proof. The homogeneous barycentric coordinates of the vertices of an I-triangle can be taken as

$$U = (u : b : c), \qquad V = (a : v : c), \qquad W = (a : b : w) \tag{1}$$

for some u, v, w. In each case, the condition for perspectivity is

$$F(u, v, w) := (b-c)vw+(c-a)wu+(a-b)uv+(a-b)(b-c)(c-a) = 0. \tag{2}$$

□

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Let D, E, F be the midpoints of the sides BC, CA, AB of triangle ABC. If $P = (x : y : z)$ is the perspector of an I-triangle UVW with the medial triangle, then U is the intersection of the line DP with the bisector IA. It has coordinates

$$((b-c)x : b(y-z) : c(y-z)).$$

Similarly, the coordinates of V and W can be determined. The triangle UVW is perspective with the intouch triangle at

$$Q = \left(\frac{x(y+z-x)}{s-a} : \frac{y(z+x-y)}{s-b} : \frac{z(x+y-z)}{s-c}\right).$$

Conversely, if an I-triangle is perspective with the intouch triangle at $Q = (x : y : z)$, then it is perspective with the medial triangle at

$$P = ((s-a)x((s-b)y + (s-c)z - (s-a)x) : \cdots : \cdots).$$

Theorem 2. *Let UVW be an I-triangle perspective with the medial and the intouch triangles. If U_1, U_1, W_1 are the inversive images of U, V, W in the incircle, then $U_1V_1W_1$ is also an I-triangle perspective with the medial and intouch triangles.*

Proof. If the coordinates of U, V, W are as given in (1), then

$$U_1 = (u_1 : b : c), \qquad V_1 = (a : v_1 : c), \qquad W_1 = (a : b : w_1),$$

where

$$\begin{aligned}
u_1 =& \frac{(a(b+c)-(b-c)^2)u - 2(s-a)(b-c)^2}{2(s-a)u - a(b+c) + (b-c)^2},\\
v_1 =& \frac{(b(c+a)-(c-a)^2)v - 2(s-b)(c-a)^2}{2(s-b)v - b(c+a) + (c-a)^2},\\
w_1 =& \frac{(c(a+b)-(a-b)^2)w - 2(s-c)(a-b)^2}{2(s-c)w - c(a+b) + (a-b)^2}.
\end{aligned}$$

From these,

$$F(u_1, v_1, w_1) = \frac{64abc(s-a)(s-b)(s-c)}{\prod_{\text{cyclic}}(2(s-a)u - a(b+c) + (b-c)^2)} \cdot F(u, v, w) = 0.$$

It follows from (2) that $U_1V_1W_1$ is perspective to both the medial and the intouch triangles. □

If an I-triangle UVW is perspective with the medial triangle at $(x : y : z)$, then $U_1V_1W_1$ is perspective with

(i) the medial triangle at

$$((y+z-x)((a(b+c)-(b-c)^2)x - (b+c-a)(b-c)(y-z)) : \cdots : \cdots),$$

(ii) the intouch triangle at

$$(a((a(b+c)-(b-c)^2)x - (b+c-a)(b-c)(y-z)) : \cdots : \cdots).$$

Theorem 3. *Let UVW be an I-triangle perspective with the medial and the intouch triangles. If U_2 (respectively V_2 and W_2) is the inversive image of U (respectively V and W) in the A- (respectively B- and C-) excircle, then $U_2V_2W_2$ is also an I-triangle perspective with the medial and intouch triangles.*

If an I-triangle UVW is perspective with the medial triangle at $(x : y : z)$, then $U_2V_2W_2$ is perspective with
(i) the medial triangle at

$$\big((s-a)(y+z-x)((a(b+c)+(b-c)^2)x+2s(b-c)(y-z)) : \cdots : \cdots\big),$$

(ii) the intouch triangle at

$$\left(\frac{a}{s-a}((a(b+c)+(b-c)^2)x+2s(b-c)(y-z)) : \cdots : \cdots\right).$$

2. Some interesting examples

We present some interesting examples of I-triangles perspective with both the medial and intouch triangles. The perspectors in these examples are new triangle centers not in the current edition of [1].

2.1. Let X_a, X_b, X_c be the inversive images of the excenters I_a, I_b, I_c in the incircle. We have $IX_a = \frac{r^2}{II_a}$ and $II_a = AI_a - AI = AI \cdot \left(\frac{s}{s-a} - 1\right) = \frac{a \cdot AI}{s-a}$. Hence,

$$\frac{AI}{IX_a} = \frac{a \cdot AI^2}{r^2(s-a)} = \frac{a}{sin^2(\frac{A}{2})} = \frac{abc}{(s-a)(s-b)(s-c)} = \frac{4R}{r},$$

and by symmetry $\frac{AI}{IX_a} = \frac{BI}{IX_b} = \frac{CI}{IX_c} = \frac{4R}{r}$. Therefore, triangles ABC and $X_aX_bX_c$ are homothetic with ratio $4R : -r$.

$$\begin{aligned}
X_a =&(a^2+b^2+c^2-2ab-2bc-2ca)(a,b,c)\\
&+(b+c-a)(c+a-b)(a+b-c)(1,0,0),\\
X_b =&(a^2+b^2+c^2-2ab-2bc-2ca)(a,b,c)\\
&+(b+c-a)(c+a-b)(a+b-c)(0,1,0),\\
X_c =&(a^2+b^2+c^2-2ab-2bc-2ca)(a,b,c)\\
&+(b+c-a)(c+a-b)(a+b-c)(0,0,1).
\end{aligned}$$

Proposition 4. *$X_aX_bX_c$ is an I-triangle perspective with*
(i) the medial triangle at

$$P_x = (a^2(b+c)+(b+c-2a)(b-c)^2 : \cdots : \cdots),$$

(ii) the intouch triangle at

$$Q_x = (a(b+c-a)(a^2(b+c)+(b+c-2a)(b-c)^2) : \cdots : \cdots).$$

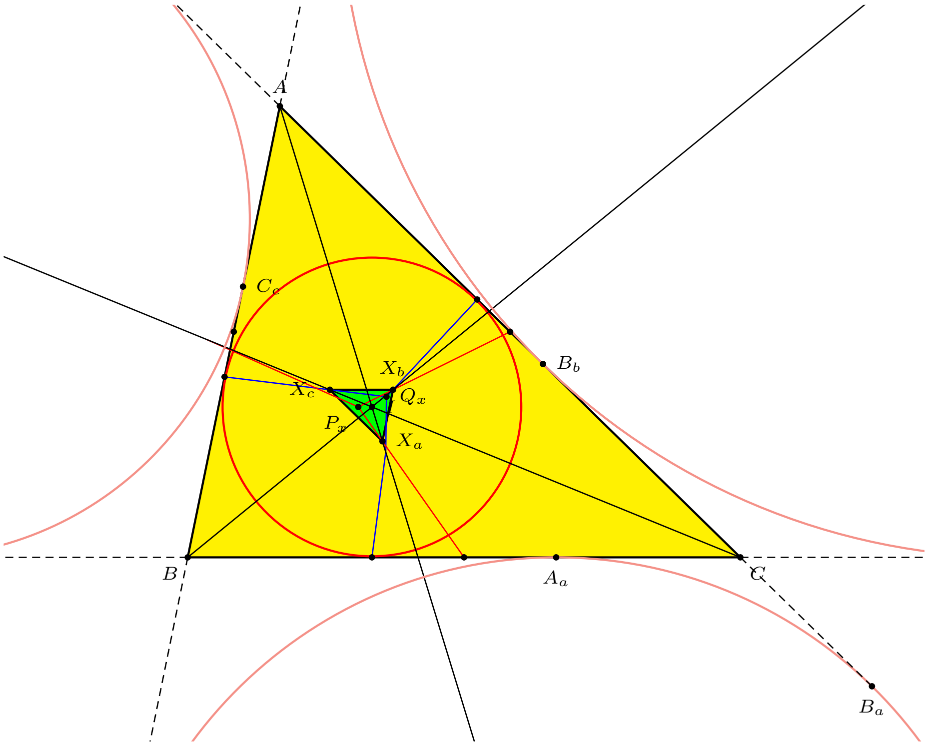

Figure 2.

2.2. Let Y_a, Y_b, Y_c be the inversive images of the incenter I with respect to the A-, B-, C-excircles.

$$\begin{aligned}
Y_a =&(a^2+b^2+c^2-2bc+2ca+2ab)(-a,b,c)\\
&+(a+b+c)(c+a-b)(a+b-c)(1,0,0),\\
Y_b =&(a^2+b^2+c^2+2bc-2ca+2ab)(a,-b,c)\\
&+(a+b+c)(a+b-c)(b+c-a)(0,1,0),\\
Y_c =&(a^2+b^2+c^2+2bc+2ca-2ab)(a,b,-c)\\
&+(a+b+c)(b+c-a)(c+a-b)(1,0,0).
\end{aligned}$$

Proposition 5. *$Y_aY_bY_c$ is an I-triangle perspective with*
(i) the medial triangle at

$$P_y=((b+c-a)^2(a^2(b+c)+(2a+b+c)(b-c)^2):\cdots:\cdots),$$

(ii) the intouch triangle at

$$Q_y=\left(\frac{a(a^2(b+c)+(2a+b+c)(b-c)^2)}{b+c-a}:\cdots:\cdots\right).$$

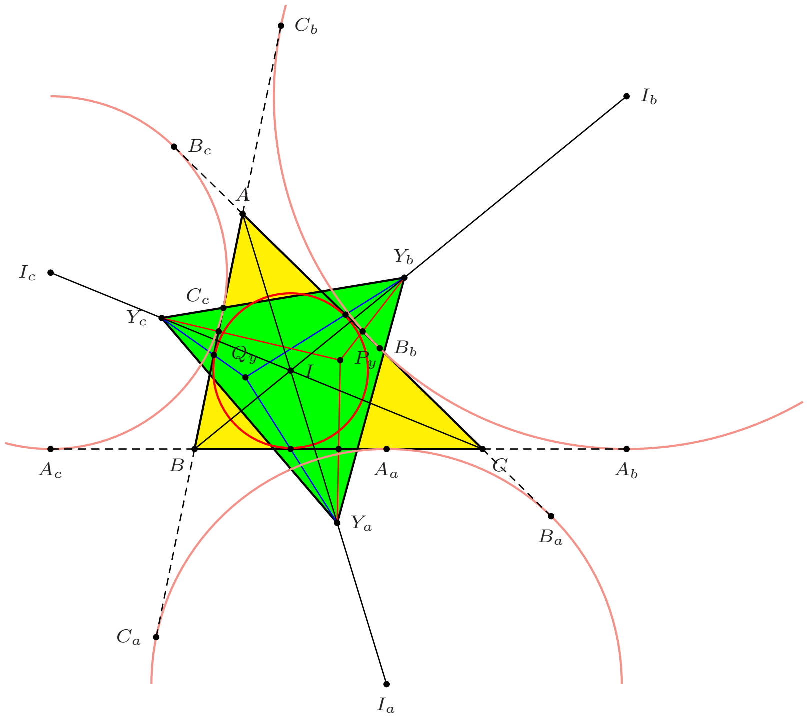

Figure 3.

2.3. Let V_a, V_b, V_c be the inversive images of X_a, X_b, X_c with respect to the A-, B-, C-excircles.

$$\begin{aligned}V_a =&(3a^2 + (b-c)^2)(-a, b, c) + 2a(c+a-b)(a+b-c)(1, 0, 0),\\ V_b =&(3b^2 + (c-a)^2)(a, -b, c) + 2b(a+b-c)(b+c-a)(0, 1, 0),\\ V_c =&(3c^2 + (a-b)^2)(a, b, -c) + 2c(b+c-a)(c+a-b)(0, 0, 1).\end{aligned}$$

Proposition 6. *$V_aV_bV_c$ is an I-triangle perspective with*
(i) the medial triangle at

$$P_v = (a(b+c-a)^3(a^2 + 3(b-c)^2) : \cdots : \cdots),$$

(ii) the intouch triangle at

$$Q_v = \left(\frac{a(a^2 + 3(b-c)^2)}{b+c-a} : \cdots : \cdots\right).$$

2.4. Let W_a, W_b, W_c be the inversive images of Y_a, Y_b, Y_c with respect to the incircle.

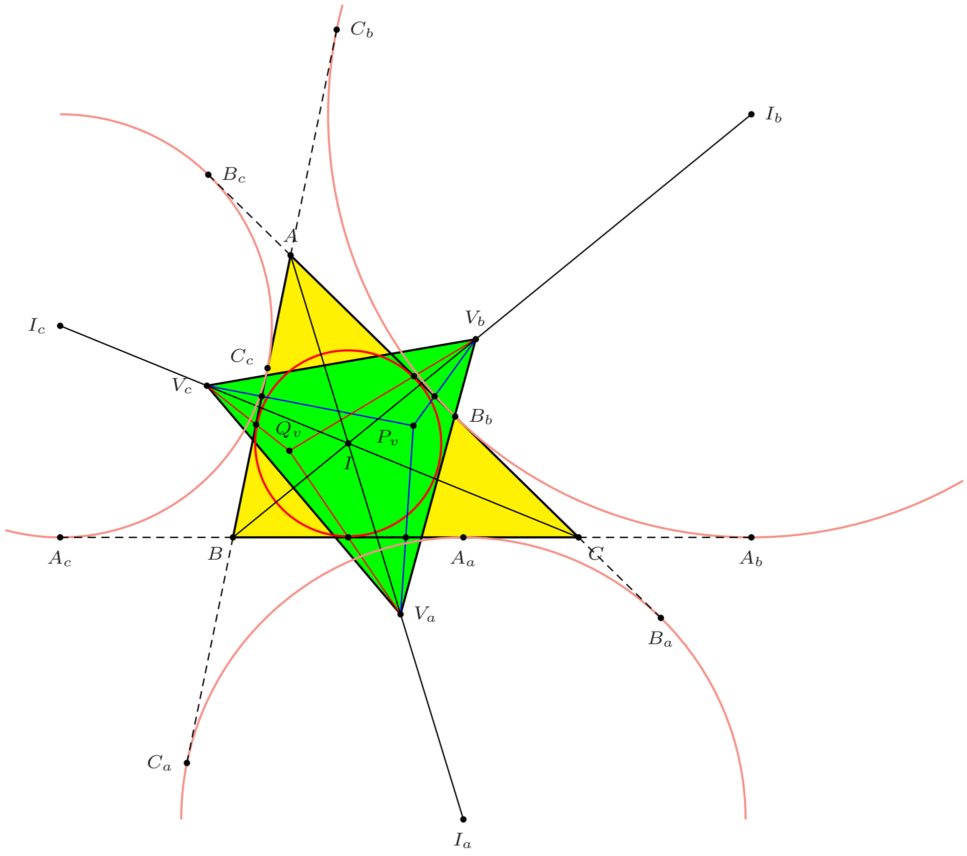

Figure 4.

$$\begin{aligned}
W_a =&(3a^2 + (b-c)^2)(a,b,c) - 2a(c+a-b)(a+b-c)(1,0,0),\\
W_b =&(3b^2 + (c-a)^2)(a,b,c) - 2b(a+b-c)(b+c-a)(0,1,0),\\
W_c =&(3c^2 + (a-b)^2)(a,b,c) - 2c(b+c-a)(c+a-b)(0,0,1).
\end{aligned}$$

Proposition 7. *$W_aW_bW_c$ is an I-triangle perspective with*
(i) the medial triangle at

$$P_w = (a(a^2 + 3(b-c)^2) : \cdots : \cdots),$$

(ii) the intouch triangle at

$$Q_w = (a(b+c-a)^2(a^2 + 3(b-c)^2) : \cdots : \cdots).$$

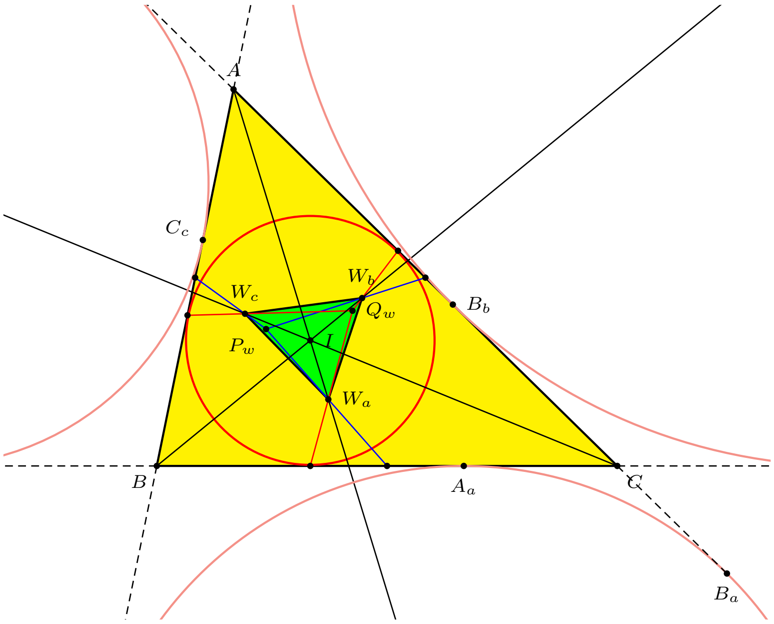

Figure 5.

Reference

[1] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

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