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Formulas Among Diagonals in the Regular Polygon and the Catalan Numbers

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Abstract. We look at relationships among the lengths of diagonals in the regular polygon. Specifically, we derive formulas for all diagonals in terms of the shortest diagonals and other formulas in terms of the next-to-shortest diagonals, assuming unit side length. These formulas are independent of the number of sides of the regular polygon. We also show that the formulas in terms of the shortest diagonals involve the famous Catalan numbers.

1. Motivation

In [1], Fontaine and Hurley develop formulas that relate the diagonal lengths of a regular n-gon. Specifically, given a regular convex n-gon whose vertices are $P_0, P_1, \ldots, P_{n-1}$, define d_k as the distance between P_0 and P_k. Then the law of sines yields

$$\frac{d_k}{d_j} = \frac{\sin \frac{k\pi}{n}}{\sin \frac{j\pi}{n}}.$$

Defining

$$r_k = \frac{\sin \frac{k\pi}{n}}{\sin \frac{\pi}{n}},$$

the formulas given in [1] are

$$r_h r_k = \sum_{i=1}^{\min\{k,h,n-k,n-h\}} r_{|k-h|+2i-1}$$

and

$$\frac{1}{r_k} = \sum_{j=1}^{s} r_{k(2j-1)}$$

where $s = \min\{j > 0 : jk \equiv \pm 1 \bmod n\}$.

Notice that for $1 \leq k \leq n-1$, $r_k = \frac{d_k}{d_1}$, but there is no *a priori* restriction on k in the definition of r_k. Thus, it would make perfect sense to consider $r_0 = 0$ and $r_{-k} = -r_k$ not to mention r_k for non-integer values of k as well. Also, the only restriction on n in the definition of r_n is that n not be zero or the reciprocal of an integer.

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2. Short proofs of r_k formulas

Using the identity $\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$, we can provide some short proofs of formulas equivalent, and perhaps simpler, to those in [1]:

Proposition 1. *For integers h and k,*

$$r_h r_k = \sum_{j=0}^{k-1} r_{h-k+2j+1}.$$

Proof. Letting h and k be integers, we have

$$\begin{aligned}
r_h r_k &= (\sin\frac{\pi}{n})^{-2}\sin\frac{h\pi}{n}\sin\frac{k\pi}{n} \\
&= (\sin\frac{\pi}{n})^{-2}\frac{1}{2}\left(\cos\frac{(h-k)\pi}{n} - \cos\frac{(h+k)\pi}{n}\right) \\
&= (\sin\frac{\pi}{n})^{-2}\left(\sum_{j=0}^{k-1}\frac{1}{2}\left(\cos\frac{(h-k+2j)\pi}{n} - \cos\frac{(h-k+2j+2)\pi}{n}\right)\right) \\
&= (\sin\frac{\pi}{n})^{-2}\sum_{j=0}^{k-1}\left(\sin\frac{(h-k+2j+1)\pi}{n}\sin\frac{\pi}{n}\right) \\
&= \sum_{j=0}^{k-1} r_{h-k+2j+1}.
\end{aligned}$$

The third equality holds since the sum telescopes. □

Note that we can switch the roles of h and k to arrive at the formula

$$r_k r_h = \sum_{j=0}^{h-1} r_{k-h+2j+1}.$$

To illustrate that this is not contradictory, consider the example when $k = 2$ and $h = 5$. From Proposition 1, we have

$$r_5 r_2 = r_4 + r_6.$$

Reversing the roles of h and k, we have

$$r_2 r_5 = r_{-2} + r_0 + r_2 + r_4 + r_6.$$

Recalling that $r_0 = 0$ and $r_{-j} = r_j$, we see that these two sums are in fact equal.

The reciprocal formula in [1] is proven almost as easily:

Proposition 2. *Given an integer k relatively prime to n,*

$$\frac{1}{r_k} = \sum_{j=1}^{s} r_{k(2j-1)}$$

where s is any (not necessarily the smallest) positive integer such that $ks \equiv \pm 1$ mod n.

Proof. Starting at the right-hand side, we have

$$\begin{aligned}
\sum_{j=1}^{s} r_{k(2j-1)} &= \left(\sin\frac{\pi}{n}\sin\frac{k\pi}{n}\right)^{-1}\sum_{j=1}^{s}\sin\frac{k(2j-1)\pi}{n}\sin\frac{k\pi}{n} \\
&= \left(\sin\frac{\pi}{n}\sin\frac{k\pi}{n}\right)^{-1}\sum_{j=1}^{s}\frac{1}{2}\left(\cos\frac{k(2j-2)\pi}{n}-\cos\frac{2jk\pi}{n}\right) \\
&= \left(\sin\frac{\pi}{n}\sin\frac{k\pi}{n}\right)^{-1}\frac{1}{2}\left(\cos 0-\cos\frac{2sk\pi}{n}\right) \\
&= \left(\sin\frac{\pi}{n}\sin\frac{k\pi}{n}\right)^{-1}\sin^2\frac{sk\pi}{n} \\
&= \left(\sin\frac{\pi}{n}\sin\frac{k\pi}{n}\right)^{-1}\sin^2\frac{\pi}{n} \\
&= \left(\sin\frac{k\pi}{n}\right)^{-1}\sin\frac{\pi}{n} \\
&= \frac{1}{r_k}.
\end{aligned}$$

Here, the third equality follows from the telescoping sum, and the fifth follows from the definition of s. □

3. From powers of r_2 to Catalan numbers

We use the special case of Proposition 1 when $h = 2$, namely

$$r_k r_2 = r_{k-1} + r_{k+1},$$

to develop formulas for powers of r_2.

Proposition 3. *For nonnegative integers m,*

$$r_2{}^m = \sum_{i=0}^{m}\binom{m}{i} r_{1-m+2i}.$$

Proof. We proceed by induction on m. When we have $m = 0$, the formula reduces to $r_2{}^0 = r_1$ and both sides equal 1. This establishes the basis step. For the inductive step, we assume the result for $m = n$ and begin with the sum on the right-hand side for $m = n + 1$.

$$\begin{aligned}
\sum_{i=0}^{n+1}\binom{n+1}{i} r_{-n+2i} &= \sum_{i=0}^{n+1}\left(\binom{n}{i}+\binom{n}{i-1}\right) r_{-n+2i} \\
&= \left(\sum_{i=0}^{n+1}\binom{n}{i-1} r_{-n+2i}\right)+\left(\sum_{i=0}^{n+1}\binom{n}{i} r_{-n+2i}\right) \\
&= \left(\sum_{j=0}^{n}\binom{n}{j} r_{2-n+2j}\right)+\left(\sum_{i=0}^{n}\binom{n}{i} r_{-n+2i}\right) \\
&= \sum_{i=0}^{n}\binom{n}{i}\left(r_{-n+2i}+r_{2-n+2i}\right) \\
&= \sum_{i=0}^{n}\binom{n}{i} r_2 r_{1-n+2i} \\
&= r_2 {r_2}^n \\
&= {r_2}^{n+1}
\end{aligned}$$

which completes the induction. The first equality uses the standard identity for binomial coefficients $\binom{n+1}{i}=\binom{n}{i}+\binom{n}{i-1}$.

The third equality is by means of the change of index $j=i-1$ and the fact that $\binom{n}{c}=0$ if $c<0$ or $c>n$.

The fifth equality is from Proposition 1 and the sixth is from the induction hypothesis. □

Now, we use Proposition 3 and the identity $r_{-k}=-r_k$ to consider an expression for ${r_2}^m$ as a linear combination of r_k's where $k>0$, i.e., we wish to determine the coefficients α_k in the sum

$${r_2}^m=\sum_{k=1}^{m+1}\alpha_{m,k} r_k.$$

From Proposition 2, the sum is known to end at $k=m+1$. In fact, we can determine α_k directly. One contribution occurs when $k=1-m+2i$, or $i=\frac{1}{2}(m+k-1)$. A second contribution occurs when $-k=1-m+2i$, or $i=\frac{1}{2}(m-k-1)$. Notice that if m and k have the same parity, there is no contribution to $\alpha_{m,k}$. Piecing this information together, we find

$$\alpha_{m,k}=\begin{cases}\binom{m}{\frac{1}{2}(m+k-1)}-\binom{m}{\frac{1}{2}(m-k-1)}, & \text{if } m-k \text{ is odd;}\\ 0, & \text{if } m-k \text{ is even.}\end{cases}$$

Notice that if $m-k$ is odd, then

$$\binom{m}{\frac{1}{2}(m-k+1)}=\binom{m}{\frac{1}{2}(m+k-1)}.$$

Therefore,

$$\alpha_{m,k} = \begin{cases} \binom{m}{\frac{1}{2}(m-k+1)} - \binom{m}{\frac{1}{2}(m-k-1)}, & \text{if } m+k \text{ is odd;} \\ 0, & \text{if } m+k \text{ is even.} \end{cases}$$

Intuitively, if we arrange the coefficients of the original formula in a table, indexed horizontally by $k \in \mathbb{Z}$ and vertically by $m \in \mathbb{N}$, then we obtain Pascal's triangle (with an extra row corresponding to $m = 0$ attached to the top):

	-2	-1	0	1	2	3	4	5	6
0	0	0	0	1	0	0	0	0	0
1	0	0	0	0	1	0	0	0	0
2	0	0	0	1	0	1	0	0	0
3	0	0	1	0	2	0	1	0	0
4	0	1	0	3	0	3	0	1	0
5	1	0	4	0	6	0	4	0	1

Next, if we subtract the column corresponding to $s = -k$ from that corresponding to $s = k$, we obtain the $\alpha_{m,k}$ array:

	1	2	3	4	5	6
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	1	0	1	0	0	0
3	0	2	0	1	0	0
4	2	0	3	0	1	0
5	0	5	0	4	0	1

As a special case of this formula, consider what happens when $m = 2p$ and $k = 1$. We have

$$\begin{aligned} \alpha_1 &= \binom{2p}{p} - \binom{2p}{p-1} \\ &= \frac{(2p)!}{p!p!} - \frac{(2p)!}{(p-1)!(p+1)!} \\ &= \frac{(2p)!}{p!p!}\left(1 - \frac{p}{p+1}\right) \\ &= \frac{1}{p+1}\binom{2p}{p} \end{aligned}$$

which is the closed form for the pth Catalan number.

4. Inverse formulas, polynomials, and binomial coefficients

We have a formula for the powers of r_2 as linear combinations of the r_k values. We now derive the inverse relationship, writing the n_k values as linear combinations of powers of r_2. We start with Proposition 1: $r_2 r_k = r_{k-1} + r_{k+1}$.

We demonstrate that for natural numbers k, there exist polynomials $P_k(t)$ such that $r_k = P_k(r_2)$. We note that $P_0(t) = 0$ and $P_1(t) = 1$. Now assume the existence of $P_{k-1}(t)$ and $P_k(t)$. Then from the identity

$$r_{k+1} = r_2 r_k - r_{k-1}$$

we have

$$P_{k+1}(t) = tP_k(t) - P_{k-1}(t)$$

which establishes a second-order recurrence for the polynomials $P_k(t)$. Armed with this, we show for $k \geq 0$,

$$P_k(t) = \sum_i \binom{k-1-i}{i} (-1)^i t^{k-1-2i}.$$

By inspection, this holds for $k = 0$ as the binomial coefficients are all zero in the sum. Also, when $k = 1$, the $i = 0$ term is the only nonzero contributor to the sum. Therefore, it is immediate that this formula holds for $k = 0, 1$. Now, we use the recurrence to establish the induction. Given $k \geq 1$,

$$\begin{aligned} P_{k+1}(t) &= tP_k(t) - P_{k-1}(t) \\ &= t\sum_i \binom{k-1-i}{i} (-1)^i t^{k-1-2i} - \sum_j \binom{k-2-j}{j} (-1)^j t^{k-2-2j} \\ &= \sum_i \binom{k-1-i}{i} (-1)^i t^{k-2i} - \sum_i \binom{k-1-i}{i-1} (-1)^{i-1} t^{k-2i} \\ &= \sum_i \binom{k-i}{i} (-1)^i t^{k-2i} \end{aligned}$$

as desired. The third equality is obtained by replacing j with $i-1$.

As a result, we obtain the desired formula for r_k in terms of powers of r_2:

Proposition 4.

$$r_k = \sum_i \binom{k-1-i}{i} (-1)^i {r_2}^{k-1-2i}.$$

Assembling these coefficients into an array similar to the $\alpha_{m,k}$ array in the previous section, we have

	0	1	2	3	4	5	6
1	1	0	0	0	0	0	0
2	0	1	0	0	0	0	0
3	−1	0	1	0	0	0	0
4	0	−2	0	1	0	0	0
5	1	0	−3	0	1	0	0
6	0	3	0	−4	0	1	0
7	−1	0	6	0	−5	0	1

This array displays the coefficient $\beta(k, m)$ in the formula

$$r_k = \sum_m \beta(k, m) r_2^m,$$

where k is the column number and m is the row number. An interesting observation is that this array and the $\alpha(m,k)$ array are inverses in the sense of "array multiplication":

$$\sum_i \alpha(m,i)\beta(i,p) = \begin{cases} 1, & m = p\,; \\ 0, & \text{otherwise.} \end{cases}$$

Also, the $k^{(th)}$ column of the $\beta(k,m)$ array can be generated using the generating function $x^k(1+x^2)^{-1-k}$. Using machinery in §5.1 of [2], this leads to the conclusion that the columns of the original $\alpha(m,k)$ array can be generated using an inverse function; in this case, the function that generates the $m^{(th)}$ column of the $\alpha(m,k)$ array is

$$x^{m-1}\left(\frac{1-\sqrt{1-4x^2}}{2x^2}\right)^m.$$

This game is similar but slightly more complicated in the case of r_3. Here, we use

$$r_3 r_k = r_{k+2} + r_k + r_{k-2}$$

which leads to

$$r_{k+2} = (r_3 - 1)r_k + r_{k-2}.$$

With this, we show that for $k \geq 0$, there are functions, but not necessarily polynomials, $Q_k(t)$ such that $r_k = Q_k(r_3)$. From the above identity, we have

$$Q_{k+2}(t) = (t-1)Q_k(t) - Q_{k-2}(t).$$

This establishes a fourth-order recurrence relation for the functions $Q_k(t)$ so determining the four functions Q_0, Q_1, Q_2, and Q_3 will establish the recurrence. By inspection, $Q_0(t) = 0$, $Q_1(t) = 1$, and $Q_3(t) = t$ so all that remains is to determine $Q_2(t)$. We have $r_3 = {r_2}^2 - 1$ from Proposition 4. Therefore, $r_2 = \sqrt{r_3+1}$ and so $Q_2(t) = \sqrt{t+1}$.

We now claim

Proposition 5. *For all natural numbers k,*

$$Q_{2k}(t) = \sqrt{t+1}\sum_i \binom{k-1-i}{i}(-1)^i(t-1)^{k-1-2i}$$

$$Q_{2k+1}(t) = \sum_i \binom{k-i}{i}(-1)^i(t-1)^{k-2i} + \sum_i \binom{k-1-i}{i}(-1)^i(t-1)^{k-1-2i}.$$

Proof. We proceed by induction on k. These are easily checked to match the functions Q_0, Q_1, Q_2, and Q_3 for $k = 0, 1$. For $k \geq 2$, we have

$$\begin{aligned}
Q_{2k}(t) &= (t-1)Q_{2(k-1)}(t) - Q_{2(k-2)}(t) \\
&= \sqrt{t+1}\left((t-1)\sum_i \binom{k-2-i}{i}(-1)^i(t-1)^{k-2-2i}\right. \\
&\qquad \left. - \sum_j \binom{k-3-j}{j}(-1)^j(t-1)^{k-3-2j}\right) \\
&= \sqrt{t+1}\left(\sum_i \left(\binom{k-2-i}{i} + \binom{k-2-i}{i-1}\right)(-1)^i(t-1)^{k-1-2i}\right) \\
&= \sqrt{t+1}\left(\sum_i \binom{k-1-i}{i}(-1)^i(t-1)^{k-1-2i}\right)
\end{aligned}$$

as desired. The third equality is obtained by replacing j with $i-1$. The proof for Q_{2k+1} is similar, treating each sum separately, and is omitted. □

As a corollary, we have

Corollary 6.

$$\begin{aligned}
r_{2k} &= \sqrt{r_3+1}\sum_i \binom{k-1-i}{i}(-1)^i(r_3-1)^{k-1-2i} \\
r_{2k+1} &= \sum_i \binom{k-i}{i}(-1)^i(r_3-1)^{k-2i} + \sum_i \binom{k-1-i}{i}(-1)^i(r_3-1)^{k-1-2i}.
\end{aligned}$$

References

[1] A. Fontaine and S. Hurley, Proof by picture: Products and reciprocals of diagonal length ratios in the regular polygon, *Forum Geom.*, 6 (2006) 97–101.

[2] H. Wilf, *Generatingfunctionology*, Academic Press, 1994.

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