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A Note on the Barycentric Square Roots of Kiepert Perspectors

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Abstract. Let P be an interior point of a given triangle ABC. We prove that the orthocenter of the cevian triangle of the barycentric square root of P lies on the Euler line of ABC if and only if P lies on the Kiepert hyperbola.

1. Introduction

In a recent Mathlinks message, the present author proposed the following problem.

Theorem 1. *Given an acute triangle ABC with orthocenter H, the orthocenter H' of the cevian triangle of $\sqrt{H}$, the barycentric square root of H, lies on the Euler line of triangle ABC.*

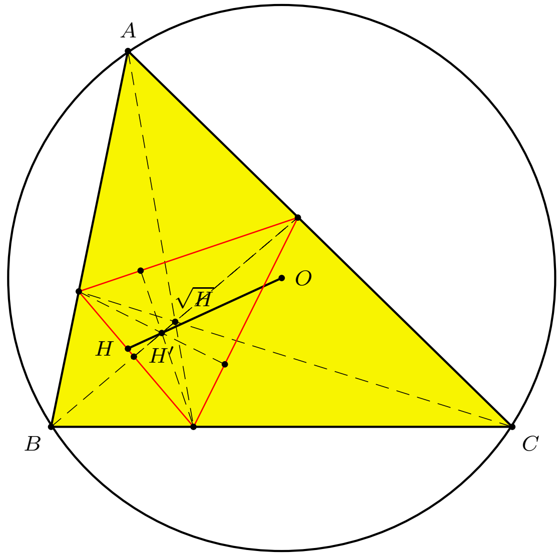

Figure 1.

Paul Yiu has subsequently discovered the following generalization.

Theorem 2. *The locus of point P for which the orthocenter of the cevian triangle of the barycentric square root $\sqrt{P}$ lies on the Euler line is the part of the Kiepert hyperbola which lies inside triangle ABC.*

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The author is grateful to Professor Yiu for his generalization of the problem and his help in the preparation of this paper.

The barycentric square root is defined only for interior points. This is the reason why we restrict to acute angled triangles in Theorem 1 and to the interior points on the Kiepert hyperola in Theorem 2. It is enough to prove Theorem 2.

2. Trilinear polars

Let $A'B'C'$ be the cevian triangle of P, and A_1, B_1, C_1 be respectively the intersections of $B'C'$ and BC, $C'A'$ and CA, $A'B'$ and AB. By Desargues' theorem, the three points A_1, B_1, C_1 lie on a line ℓ_P, the trilinear polar of P.

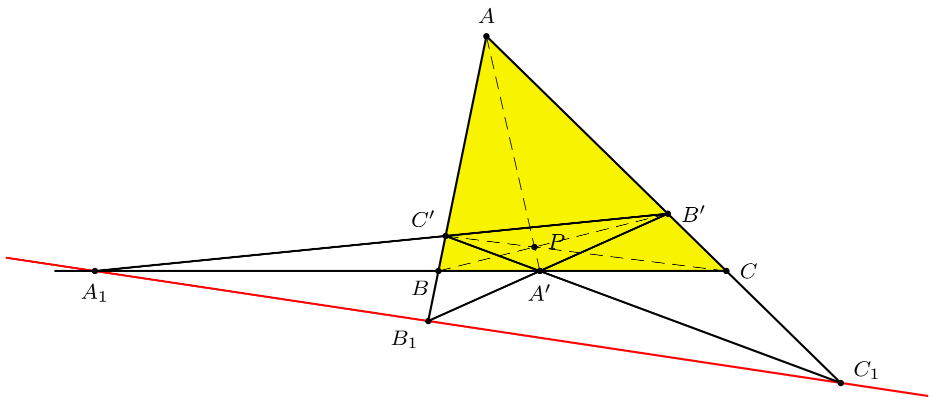

Figure 2.

If P has homogeneous barycentric coordinates $(u : v : w)$, then the trilinear polar is the line

$$\ell_P : \qquad \frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0.$$

For the orthocenter $H = (S_{BC} : S_{CA} : S_{AB})$, the trilinear polar

$$\ell_H : \qquad S_A x + S_B y + S_C z = 0.$$

is also called the orthic axis.

Proposition 3. *The orthic axis is perpendicular to the Euler line.*

This proposition is very well known. It follows easily, for example, from the fact that the orthic axis ℓ_H is the radical axis of the circumcircle and the nine-point circle. See, for example, [2, §§5.4,5].

The trilinear polar ℓ_P and the orthic axis ℓ_H intersect at the point

$$(u(S_B v - S_C w) : v(S_C w - S_A u) : w(S_A u - S_B v)).$$

In particular, ℓ_P and ℓ_H are parallel, i.e., their intersection is a point at infinity if and only if

$$u(S_B v - S_C w) + v(S_C w - S_A u) + w(S_A u - S_B v) = 0.$$

Equivalently,

$$(S_B - S_C)vw + (S_C - S_A)wu + (S_A - S_B)uv = 0. \qquad (1)$$

Note that this equation defines the Kiepert hyperbola. Points on the Kiepert hyperbola are called Kiepert perspectors.

Proposition 4. *The trilinear polar ℓ_P is parallel to the orthic axis if and only if P is a Kiepert perspector.*

3. The barycentric square root of a point

Let P be a point inside triangle ABC, with homogeneous barycentric coordinates $(u : v : w)$. We may assume $u, v, w > 0$, and define the barycentric square root of P to be the point $\sqrt{P}$ with barycentric coordinates $(\sqrt{u} : \sqrt{v} : \sqrt{w})$.

Paul Yiu [2] has given the following construction of $\sqrt{P}$.

(1) Construct the circle $\mathcal{C}_A$ with BC as diameter.

(2) Construct the perpendicular to BC at the trace A' of P to intersect $\mathcal{C}_A$ at X'.

(3) Construct the bisector of angle $BX'C$ to intersect BC at X.

Then X is the trace of $\sqrt{P}$ on BC. Similar constructions on the other two sides give the traces Y and Z of $\sqrt{P}$ on CA and AB respectively. The barycentric square root $\sqrt{P}$ is the common point of AX, BY, CZ.

Proposition 5. *If the trilinear polar ℓ_P intersects BC at A_1, then*

$$A_1X^2 = A_1B \cdot A_1C.$$

Proof. Let M is the midpoint of BC. Since A_1, A' divide B, C harmonically, we have $MB^2 = MC^2 = MA_1 \cdot MA'$ (Newton's theorem). Thus, $MX'^2 = MA_1 \cdot MA'$. It follows that triangles $MX'A_1$ and $MA'X'$ are similar, and $\angle MX'A_1 = \angle MA'X' = 90°$. This means that A_1X' is tangent at X' to the circle with diameter BC. Hence, $A_1X'^2 = A_1B \cdot A_1C$.

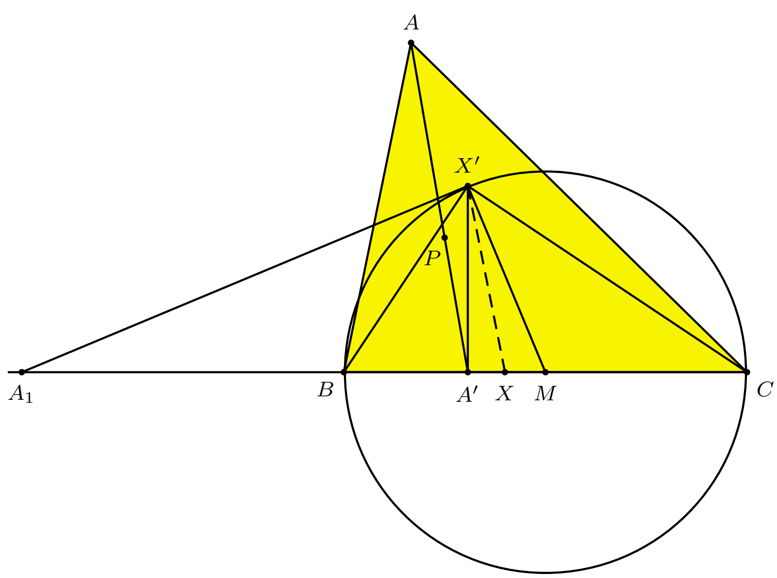

Figure 3.

To complete the proof it is enough to show that $A_1X = A_1X'$, *i.e.*, triangle A_1XX' is isosceles. This follows easly from

$$\begin{aligned}\angle A_1X'X =&\angle A_1X'B + \angle BX'X\\ =&\angle X'CB + \angle XX'C\\ =&\angle X'XA_1.\end{aligned}$$

□

Corollary 6. *If X_1 is the intersection of YZ and BC, then A_1 is the midpoint of XX_1.*

Proof. If X_1 is the intersection of YZ and BC, then X, X_1 divide B, C harmonically. The circle through X, X_1, and with center on BC is orthogonal to the circle $\mathcal{C}_A$. By Proposition 5, this has center A_1, which is therefore the midpoint of XX_1. □

4. Proof of Theorem 2

Let P be an interior point of triangle ABC, and XYZ the cevian triangle of its barycentric square root $\sqrt{P}$.

Proposition 7. *If H' is the orthocenter of XYZ, then the line OH' is perpendicular to the trilinear polar ℓ_P.*

Proof. Consider the orthic triangle DEF of XYZ. Since $DEXY$, $EFYZ$, and $FDZX$ are cyclic, and the common chords DX, EY, FZ intersect at H', H' is the radical center of the three circles, and

$$H'D \cdot H'X = H'E \cdot H'Y = H'F \cdot H'Z. \tag{2}$$

Consider the circles ξ_A, ξ_B, ξ_C, with diameters XX_1, YY_1, ZZ_1. These three circles are coaxial; they are the generalized Apollonian circles of the point $\sqrt{P}$. See [3]. As shown in the previous section, their centers are the points A_1, B_1, C_1 on the trilinear polar ℓ_P. See Figure 4.

Now, since D, E, F lie on the circles ξ_A, ξ_B, ξ_C respectively, it follows from (2) that H' has equal powers with respect to the three circles. It is therefore on the radical axis of the three circles.

We show that the circumcenter O of triangle ABC also has the same power with respect to these circles. Indeed, the power of O with respect to the circle ξ_A is

$$A_1O^2 - A_1X^2 = OA_1^2 - R^2 - A_1X^2 + R^2 = A_1B \cdot A_1C - A_1X^2 + R^2 = R^2$$

by Proposition 5. The same is true for the circles ξ_B and ξ_C. Therefore, O also lies on the radical axis of the three circles. It follows that the line OH' is the radical axis of the three circles, and is perpendicular to the line ℓ_P which contains their centers. □

The orthocenter H' of XYZ lies on the Euler line of triangle ABC if and only if the trilinear polar ℓ_P is parallel to the Euler line, and hence parallel to the orthic axis by Proposition 3. By Proposition 4, this is the case precisely when P lies on the Kiepert hyperbola. This completes the proof of Theorem 2.

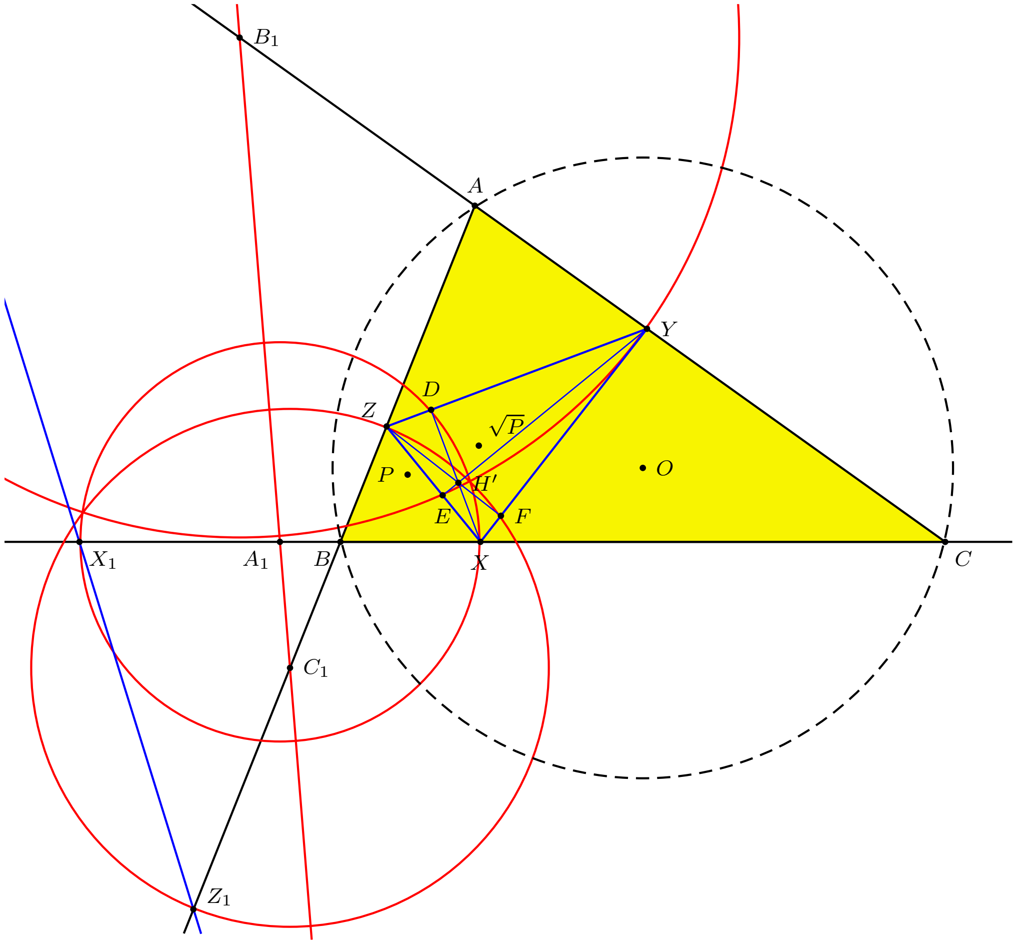

Figure 4.

Theorem 8. *The orthocenter of the cevian triangle of* $\sqrt{P}$ *lies on the Brocard axis if and only if* P *is an interior point on the Jerabek hyperbola.*

Proof. The Brocard axis OK is orthogonal to the Lemoine axis. The locus of points whose trilinear polars are parallel to the Brocard axis is the Jerabek hyperbola. □

5. Coordinates

In homogeneous barycentric coordinates, the orthocenter of the cevian triangle of $(u : v : w)$ is the point

$$\left(\left(S_B\left(\frac{1}{w}+\frac{1}{u}\right)+S_C\left(\frac{1}{u}+\frac{1}{v}\right)\right)\left(-S_A\left(\frac{1}{v}+\frac{1}{w}\right)^2+S_B\left(\frac{1}{u^2}-\frac{1}{w^2}\right)+S_C\left(\frac{1}{u^2}-\frac{1}{v^2}\right)\right)\right.$$
$$\left.:\cdots:\cdots\right).$$

Applying this to the square root of the orthocenter, with $(u^2 : v^2 : w^2) = \left(\frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C}\right)$, we obtain

$$\left(a^2 S_A \cdot \sqrt{S_{ABC}} + S_{BC}\sum_{\text{cyclic}} a^2\sqrt{S_A} : \cdots : \cdots\right),$$

which is the point H' in Theorem 1.

More generally, if P is the Kiepert perspector

$$K(\theta) = \left(\frac{1}{S_A+S_\theta} : \frac{1}{S_B+S_\theta} : \frac{1}{S_C+S_\theta}\right),$$

the orthocenter of the cevian triangle of $\sqrt{P}$ is the point

$$\left(a^2 S_A\sqrt{(S_A+S_\theta)(S_B+S_\theta)(S_C+S_\theta)}\right.$$
$$\left.+S_{BC}\sum_{\text{cyclic}} a^2\sqrt{S_A+S_\theta} + a^2 S_\theta \sum_{\text{cyclic}} S_A\sqrt{S_A+S_\theta} : \cdots : \cdots\right).$$

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[1] R. A. Johnson, *Advanced Euclidean Geometry*, 1925, Dover reprint.

[2] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University lecture notes, 2001.

[3] P. Yiu, Generalized Apollonian circles, *Journal of Geometry and Graphics*, 8 (2004) 225–230.

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