Forum Geometricorum
Volume 6 (2006) 301–310.

FORUM GEOM
ISSN 1534-1178

Two Brahmagupta Problems

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Abstract. D. E. Smith reproduces two problems from Brahmagupta's work Kutakhādyaka (algebra) in his *History of Mathematics*, Volume 1. One of them involves a broken tree and the other a mountain journey. Normally such objects are represented by vertical line segments. However, it is every day experience that such objects need not be vertical. In this paper, we generalize these situations to include slanted positions and provide integer solutions to these problems.

1. Introduction

School textbooks on geometry and trigonometry contain problems about trees, poles, buildings, hills etc. to be solved using the Pythagorean theorem or trigonometric ratios. The assumption is that such objects are vertical. However, trees grow not only vertically (and tall offering a magestic look) but also assume slanted positions (thereby offering an elegant look). Buildings too need not be vertical in structure, for example the leaning tower of Pisa. Also, a distant planar view of a mountain is more like a scalene triangle than a right one. In this paper we regard the angles formed in such situations as having rational cosines. We solve the following Brahmagupta problems from [5] in the context of rational cosines triangles. In [4] these problems have been given Pythagorean solutions.

Problem 1. A bamboo 18 cubits high was broken by the wind. Its tip touched the ground 6 cubits from the root. Tell the lengths of the segments of the bamboo.

Problem 2. On the top of a certain hill live two ascetics. One of them being a wizard travels through the air. Springing from the summit of the mountain he ascends to a certain elevation and proceeds by an oblique descent diagonally to neighboring town. The other walking down the hill goes by land to the same town. Their journeys are equal. I desire to know the distance of the town from the hill and how high the wizard rose.

We omit the numerical data given in Problem 1 to extend it to an indeterminate one like the second so that an infinity of integer solutions can be found.

Publication Date: December 4, 2006. Communicating Editor: Paul Yiu.
The author thanks the referee for his suggestions.

2. Background material

An angle θ is a rational cosine angle if $\cos\theta$ is rational. If both $\cos\theta$ and $\sin\theta$ are rational, then θ is called a Heron angle. If the angles of a triangle are rational cosine (respectively Heron) angles, then the sides are rational in proportion, and they can be rendered integers, by after multiplication by the lcm of the denominators. Thus, in effect, we deal with triangles of integer sides. Given a rational cosine (respectively Heron) angle θ, it is possible to determine the infinite family of integer triangles (respectively Heron triangles) in which each member triangle contains θ. Our discussion depends on such families of triangles, and we give the following description.

2.1. *Integer triangle family containing a given rational cosine angle θ.* Let $\cos\theta = \lambda$ be a rational number. When θ is obtuse, λ is negative. Our discussion requires that $0 < \theta < \frac{\pi}{2}$ so we must have $0 < \lambda < 1$. Let ABC be a member triangle in which $\angle BAC = \theta$. Let $\angle ABC = \phi$ as shown in Figure 1.

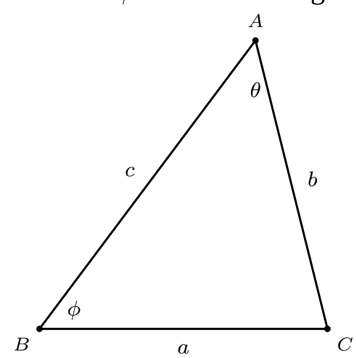

Figure 1

Applying the law of cosines to triangle ABC we have $a^2 = b^2 + c^2 - 2bc\lambda$, or

$$(c-a)(c+a) = b(2\lambda c - b).$$

By the triangle inequality $c - a < b$ so that

$$1 > \frac{c-a}{b} = \frac{2\lambda c - b}{c+a} = \frac{v}{u},$$

say, with $\gcd(u, v) = 1$. We then solve the resulting simultaneous equations

$$c - a = \frac{v}{u}a, \qquad c + a = \frac{u}{v}(2\lambda c - b)$$

for a, b, c in proportional values:

$$\frac{a}{u^2 - 2\lambda uv + v^2} = \frac{b}{2(\lambda u - v)} = \frac{c}{u^2 - v^2}.$$

We replace λ by a rational number $\frac{n}{m}$, $0 < \frac{n}{m} < 1$, and obtain a parametrization of triangles in the $\theta = \arccos\frac{n}{m}$ family:

$$(a, b, c) = (m(u^2 + v^2) - 2nuv,\ 2u(nu - mv),\ m(u^2 - v^2)), \qquad \frac{u}{v} > \frac{m}{n}. \quad (\dagger)$$

It is routine to check that

$$\phi = \arccos \frac{mc - nb}{ma},$$

and that $\cos A = \frac{n}{m}$ independently of the parameters u, v of the family described in (†) above. Here are two particular integer triangle families.

(1) The $\frac{\pi}{3}$ integer family is given by (†) with $n = 1$, $m = 2$:

$$(a, b, c) = (u^2 - uv + v^2,\ u(u - 2v),\ u^2 - v^2), \quad u > 2v,\ \gcd(u, v) = 1. \quad (1)$$

It is common practice to list primitive solutions except under special circumstances. In (1) we have removed $\gcd(a, b, c) = 2$.

(2) When θ is a Heron angle, *i.e.*, $\cos\theta = \frac{p^2-q^2}{p^2+q^2}$ for integers p, q with $\gcd(p, q) = 1$, (†) describes a Heron triangle family. For example, with $p = 2$, $q = 1$, we have $\cos\theta = \frac{3}{5}$. Now with $n = 3$, $m = 5$, (†) yields

$$(a, b, c) = (5u^2 - 6uv + 5v^2,\ 2u(3u - 5v),\ 5(u^2 - v^2)), \quad \gcd(u, v) = 1. \quad (2)$$

This has area $\triangle = \frac{1}{2}bc\sin\theta = \frac{2}{5}bc$. We may put $u = 3$, $v = 1$ to obtain the specific Heron triangle $(a, b, c) = (4, 3, 5)$ that is Pythagorean. On the other hand, $u = 4$, $v = 1$ yields the non-Pythagorean triangle $(a, b, c) = (61, 56, 75)$ with area $\triangle = 1680$.

3. Generalization of the first problem

3.1. *Restatement.* Throughout this discussion an integer tree is one with the following properties.
(i) It has an integer length $AB = c$.
(ii) It makes a rational cosine angle ϕ with the horizontal.
(iii) When the wind breaks it at a point D the broken part $AD = d$ and the unbroken part $BD = e$ both have integer lengths.
(iv) The top A of the tree touches the ground at C at an integer length $BC = a$.
All the triangles in the configuration so formed have integer sidelengths. See Figure 2.

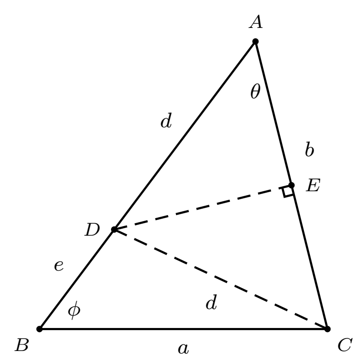

Figure 2

We note from triangle BDC, $BD + DC > BC$, *i.e.*, $AB > BC$. When $AB = BC$ the entire tree falls to the ground. Furthermore, when θ (and therefore

ϕ) is a Heron angle, AB is a Heron tree and Figure 2 represents a Heron triangle configuration.

In the original Problem 1, $c = 18$, $a = 6$ and ϕ is implicitly given (or assumed) to be $\frac{\pi}{2}$. In other words, these elements uniquely determine triangle ABC. Then the breaking point D on AB can be located as the intersection of ED, the perpendicular bisector of AC. Moreover, the present restatement of Problem 1, *i.e.*, the determination of the configuration of Figure 2, gives us an option to use either ϕ or θ as the rational cosine angle to determine triangle ABC and hence the various integer lengths $a, b, \ldots, e$. We achieve this goal with the help of $(\dagger)$. Before dealing with the general solution of Problem 1, we consider some interesting examples.

3.2. *Examples.*

3.2.1. If heavy winds break an integer tree AB at D so that the resulting configuration is an isosceles triangle with $AB = AC$, then the length of the broken part is the cube of an integer.

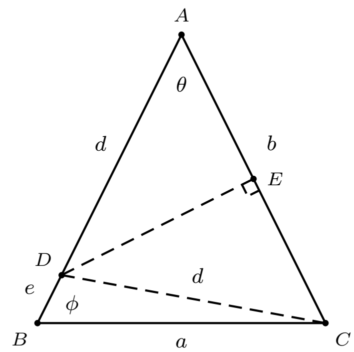

Figure 3

Proof. Suppose $AB = AC = \ell$ and $BC = m$ to begin with. From Figure 3, it follows that $\ell > m$, $\cos\theta = \frac{2\ell^2-m^2}{2\ell^2}$, $\cos\phi = \frac{m}{2\ell}$, $AD = \frac{\ell}{2\cos\theta} = \frac{\ell^3}{2\ell^2-m^2}$, $BD = \ell - \frac{\ell^3}{2\ell^2-m^2} = \frac{\ell(\ell^2-m^2)}{2\ell^2-m^2}$. to obtain integer values we multiply each by $2\ell^2 - m^2$. In the notation of Figure 2, the solution is given by

(i) $c =$ the length of the tree $= \ell(2\ell^2 - m^2)$;

(ii) $d =$ the broken part $= \ell^3$, *an integer cube*;

(iii) $e =$ the unbroken part = $\ell(\ell^2 - m^2)$;

(iv) $a =$ the distance between the foot and top of the tree $= m(2\ell^2 - m^2)$;

(v) $\phi =$ the inclination of the tree with the ground $= \arccos\frac{m}{2\ell}$. □

In particular, if $\ell = p^2 + q^2$, and $m = 2(p^2 - q^2)$ for $(\sqrt{2}+1)q > p > q$, then AB becomes a Heron tree broken by the wind. These yield

$$\begin{aligned} c = b =& 2(p^2+q^2)(2pq+p^2-q^2)(2pq-p^2+q^2),\\ d =& (p^2+q^2)^3,\\ e =& (p^2+q^2)(-p^2+3q^2)(3p^2-q^2),\\ a =& 4(p^2-q^2)(2pq+p^2-q^2)(2pq-p^2+q^2). \end{aligned}$$

For a numerical example, we put $p = 3$, $q = 2$. This gives a Heron tree of length 3094 broken by the wind into $d = 2197 = 13^3$ (an integer cube), and $e = 897$ to come down at $a = 2380$. The angle of inclination of the tree with the horizontal is $\phi = \arccos \frac{5}{13}$.

We leave the details of the following two examples to the reader as an exercise.

3.2.2. If the wind breaks an integer tree AB at D in such a way that $AC = BC$, then both the lengths of the tree and the broken part are perfect squares.

3.2.3. If the wind breaks an integer tree AB at D in such a way that $AD = DC = BC$, then the common length is a perfect square.

4. General solution of Problem 1

Ideally, the general solution of Problem 1 involves the use of integral triangles given by (†). For simplicity we first consider a special case of (†) in which $\theta = \frac{\pi}{3}$. The solution in this case is elegant. Then we simply state the general solution leaving the details to the reader.

4.1. *A particular case of Problem 1.* An integral tree AB is broken by the wind at D. The broken part DA comes down so that the top A of the tree touches the ground at C. If $\angle DAC = \frac{\pi}{3}$, determine parametric expressions for the various elements of the configuration formed.

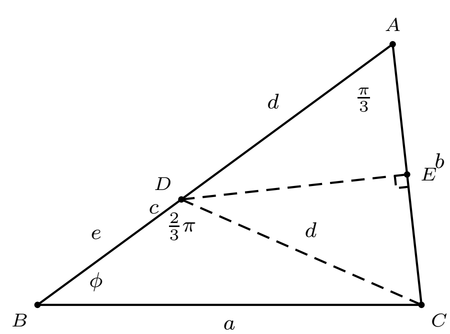

Figure 4

We refer to Figure 4. Since $\angle DAC = \frac{\pi}{3}$, triangle ADC is equilateral and $d = b = DC$. From (1), we have

$$\begin{aligned} a =& u^2 - uv + v^2, \\ d = b =& u(u-2v), \\ c =& u^2 - v^2, \\ e =& c - d = v(2u - v), \\ \phi =& \arccos\frac{2c-b}{2a} = \arccos\frac{u^2+2uv-2v^2}{2(u^2-uv+v^2)}, \quad u > 2v. \end{aligned}$$

For a specific numerical example, we take $u = 5$, $v = 2$, and obtain a tree of length $c = 21$, broken into $d = b = 5$, $e = 16$ and $a = 19$, inclined at an angle $\phi = \arccos\frac{37}{38}$.

Remark. No tree in the $\frac{\pi}{3}$ family can be Heron because $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ is irrational.

Note also that in Figure 4, $\angle BDC = \frac{2}{3}\pi$. Hence the family of triangles

$$(a, b, e) = (u^2 - uv + v^2,\ u(u - 2v),\ v(2u - v))$$

contains the angle $\frac{2}{3}\pi$ in each member. For example, with $u = 5$, $v = 2$, we have $(a, b, c) = (19, 5, 16)$; $\cos A = -\frac{1}{2}$ and $\angle A = \frac{2}{3}\pi$.

4.2. *General solution of Problem 1.* Let $\cos\theta = \frac{n}{m}$. The corresponding integral trees have
(i) length $c = mn(u^2 - v^2)$,
(ii) broken part $d = mu(nu - mv)$,
(iii) unbroken part $e = mv(mu - nv)$,
(iv) distance between the foot and the top of the tree on the ground

$$a = n(m(u^2 + v^2) - 2nuv),$$

and
(v) the angle of inclination with the ground ϕ where

$$\cos\phi = \frac{(m^2 - 2n^2)u^2 + 2mnuv - m^2v^2}{ma}. \tag{3}$$

Remark. The solution in (3) yields the solution of the Heron tree problem broken by the wind when θ is a Heron angle, *i.e.*, when $\cos\theta = \frac{p^2-q^2}{p^2+q^2}$. Here is a numerical example. Suppose $p = 2$, $q = 1$. Then $\cos\theta = \frac{3}{5}$, *i.e.*, $n = 3$, $m = 5$. To break a specific Heron tree of this family, we put $u = 4$, $v = 1$. Then we find that $c = 225$, $d = 140$, $e = 85$, $a = 183$, and the angle of inclination $\phi = \arccos\frac{207}{305}$, a Heron angle.

5. The second problem

Brahmagupta's second problem does not need any restatement. It is an indeterminate one in its original form.

An integral mountain is one whose planar view is an integral triangle. If the angles of this integral triangle are Heron angles, then the plain view becomes a heron triangle. In such a case we have a Heron mountain. One interesting feature of the second problem is the pair of integral triangles that we are required to generate for its solution. Furthermore, it creates an amusing situation as we shall soon see – in a sense there is more wizardry!

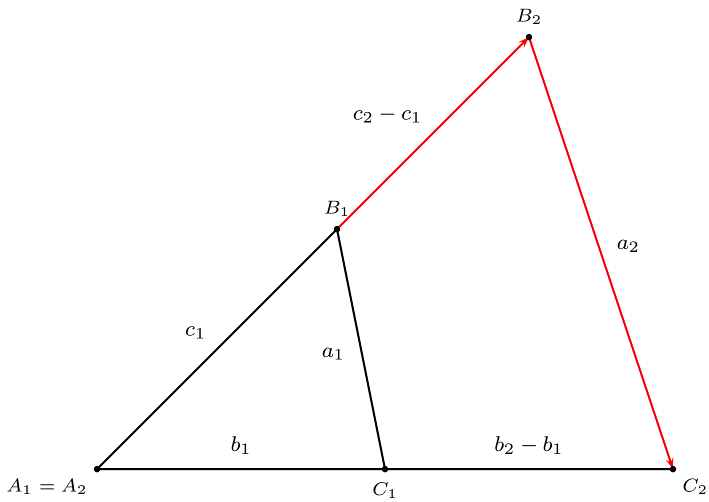

Figure 5

Figure 5 shows an integral mountain $A_1B_1C_1$. At B_1 live two ascetics. The wizard of them flies to B_2 along the direction of A_1B_1, and then reaches the town C_2. The other one walks along the path $B_1C_1C_2$. The hypothesis of the problem is $B_1B_2 + B_2C_2 = B_1C_1 + C_1C_2$, *i.e.*,

$$c_2 + a_2 - b_2 = c_1 + a_1 - b_1. \tag{4}$$

Hence the solution to Problem 2 lies in generating a pair of integral triangles $A_1B_1C_1$ and $A_2B_2C_2$, with
(i) $A_2 = A_1$,
(ii) $\angle B_1A_1C_1 = \angle B_2A_2C_2$,
(iii) $c_2 + a_2 - b_2 = c_1 + a_1 - b_1$. As the referee pointed out, together the two conditions above imply that triangles $A_1B_1C_1$ and $A_2B_2C_2$ have a common excircle opposite to the vertices C_1, C_2. Furthermore, we need integral answers to the questions
(i) the distance between the hill and the town $C_1C_2 = b_2 - b_1$,
(ii) the height the wizard rose, *i.e.*, the altitude $c_2 \sin A_1$ through B_2 of triangle $A_2B_2C_2$.

Now, if $c_2 \sin A_1$ is to be an integer, then $\sin A_1$ should necessarily be rational. Therefore the integral mountain must be a Heron mountain. We may now put $\cos\theta = \frac{p^2-q^2}{p^2+q^2}$, *i.e.*, $n = p^2 - q^2$, $m = p^2 + q^2$ in (†) to find the answers. As it turns out, the solution would not be elegant. Instead, we use the following description of the family of Heron triangles, each member triangle containing θ. This description has previously appeared in this journal [4] so we simply state the description.

Let $\cos A = \frac{p^2-q^2}{p^2+q^2}$. The Heron triangle family determining the common angle A is given by

$$\begin{aligned}(a,b,c) =&(pq(u^2+v^2),\ (pu-qv)(qu+pv),\ (p^2+q^2)uv),\\ &(u,v)=(p,q)=1,\ p\geq 1 \text{ and } pu>qv.\end{aligned} \tag{5}$$

In particular, we note that
(i) $p=q \Rightarrow A=\frac{\pi}{2}$ and $(a,b,c)=(u^2+v^2,\ u^2-v^2,\ 2uv)$, and
(ii) $(p,q)=(u,v) \Rightarrow (a,b,c)=(u^2+v^2,\ 2(u^2-v^2),\ u^2+v^2)$
are respectively the Pythagorean triangle family and the isoceles Heron triangle family.

5.1. *The solution of Problem 2.* We continue to refer to Figure 5. Since $\angle B_1A_1C_1 = \angle B_2A_2C_2$, p and q remain the same in (5). This gives

$$\begin{aligned}(a_1,b_1,c_1) =&(pq(u_1^2+v_1^2),\ (pu_1-qv_1)(qu_1+pv_1),\ (p^2+q^2)u_1v_1),\\ (a_2,b_2,c_2) =&(pq(u_2^2+v_2^2),\ (pu_2-qv_2)(qu_2+pv_2),\ (p^2+q^2)u_2v_2).\end{aligned}$$

Next, $c_2+a_2-b_2 = c_1+a_1-b_1$ simplifies to

$$v_2(qu_2+pv_2) = v_1(qu_1+pv_1) = \lambda,\ \text{a constant}. \tag{6}$$

For given p, q, there are four variables u_1, v_1, u_2, v_2, and they generate an infinity of solutions satisfying the equation (6). We now obtain two particular, numerical solutions.

5.2. *Numerical examples.* (1) We put $p=2$, $q=1$, $u_1=3$, $v_1=2$ in (6). This gives $v_2(u_2+2v_2)=14=\lambda$. It is easy to verify that $u_2=12$, $v_2=1$ is a solution. Hence we have

$$(a_1,b_1,c_1)=(13,14,15) \quad \text{and} \quad (a_2,b_2,c_2)=(145,161,30).$$

Note that $\gcd(a_ib_i,c_i)=2$, $i=1,2$, has been divided out. It is easy to verify that $c_i+a_i-b_i=14$, $i=1,2$. The answers to the questions are
(i) the distance between the hill and th twon, $b_2-b_1=147$.
(ii) The wizard rose to a height $c_2\sin A_1 = 30\times\frac{4}{5}=24$.

In fact it is possible to give as many solutions (a_i,b_i,c_i) to (6) as we wish: we have just to take sufficiently large values for λ. This creates an amusing situation as we see below.

(2) Suppose $\lambda = 2\times 3\times 5\times 7 = 210$. Then (6) becomes $v_i(u_i+2v_i)=210$. The indexing of the six solutions below is unconventional in the interest of Figure 6.

i	v_i	u_i	a_i	b_i	c_i
6	1	208	43265	43575	520
5	2	101	10205	10500	505
4	3	64	4105	4375	480
3	5	32	1049	1239	400
2	6	23	565	700	345
1	7	16	305	375	280

It is easy to check that for all six Heron triangles $c_i + a_i - b_i = 210$, $i = 1, 2, \ldots, 6$. From this we deduce that
(i) $B_iC_i + C_iC_{i+1} = B_iB_{i+1} + B_{i+1}C_{i+1}$ and
(ii) $B_iC_i + C_iC_j = B_iB_j + B_jC_j$, $i, j = 1, 2, 3, 4, 5$, $j > i$.

In other words, the two ascetics may choose to live at any of the places B_1, B_2, B_3, B_4, B_5. Then they may choose to reach any next town C_2, C_3, C_4, C_5, C_6. See Figure 6.

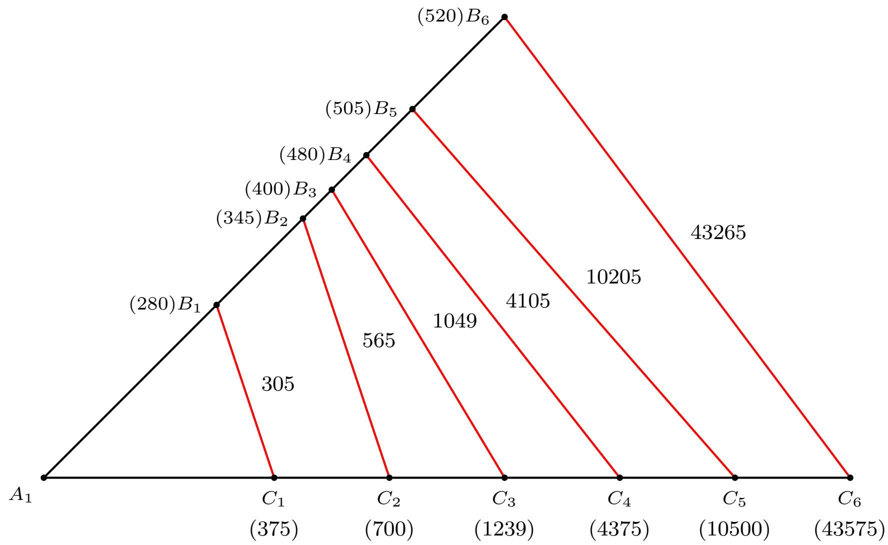

Figure 6

Another famous problem, ladders leaning against vertical walls, has been solved in the context of Heron triangles in [2].

References

[1] L. E. Dickson, *History of the Theory of Numbers*, volume II, Chelsea, New York, 1971; pp.165–224.
[2] K. R. S. Sastry, Heron ladders, *Math. Spectrum*, 34 (2001-2002) 61–64.
[3] K. R. S. Sastry, Construction of Brahmagupta n-gons, *Forum Geom.*, 5 (2005)119–126.
[4] K. R. S. Sastry, Brahmagupta's problems, Pythaogrean solutions, and Heron triangles, *Math. Spectrum*, 38 (2005-2006) 68–73.
[5] D. E. Smith, *History of Mathematics*, volume 1, Dover, 1958 pp.152–160.

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