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Square Wreaths Around Hexagons

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Abstract. We investigate the figures that arise when squares are attached to a triple of non-adjacent sides of a hexagon, and this procedure is repeated with alternating choice of the non-adjacent sides. As a special case we investigate the figure that starts with a triangle.

1. Square wreaths around hexagons

Consider a hexagon $\mathcal{H}_1 = H_{1,1}H_{2,1}H_{3,1}H_{4,1}H_{5,1}H_{6,1}$ with counterclockwise orientation. We attach squares externally on the sides $H_{1,1}H_{2,1}$, $H_{3,1}H_{4,1}$ and $H_{5,1}H_{6,1}$, to form a new hexagon $\mathcal{H}_2 = H_{1,2}H_{2,2}H_{3,2}H_{4,2}H_{5,2}H_{6,2}$. Following Nottrot, [8], we say we have made the first *square wreath* around $\mathcal{H}_1$. Then we attach externally squares to the sides $H_{6,2}H_{1,2}$, $H_{2,2}H_{3,2}$ and $H_{4,2}H_{5,2}$, to get a third hexagon $\mathcal{H}_3$, creating the second square wreath. We may repeat this operation to find a sequence of hexagons $\mathcal{H}_n = H_{1,n}H_{2,n}H_{3,n}H_{4,n}H_{5,n}H_{6,n}$ and square wreaths. See Figure 1.

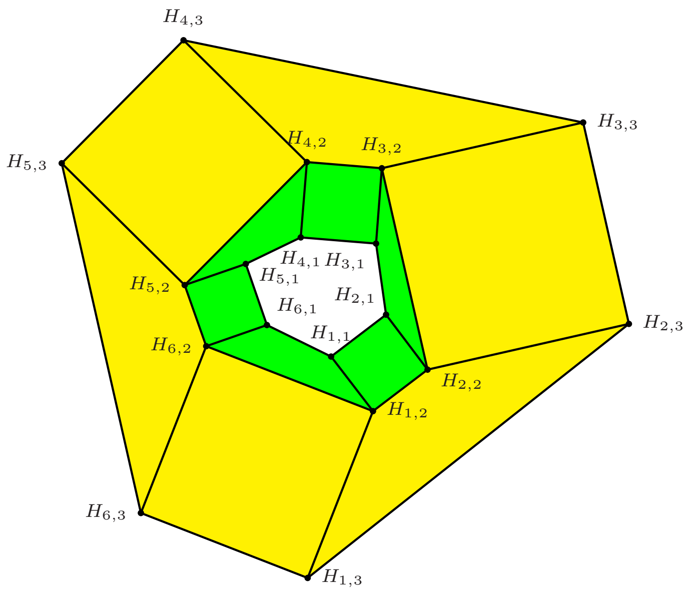

Figure 1

We introduce complex number coordinates, and abuse notations by identifying a point with its affix. Thus, we shall also regard $H_{m,n}$ as a complex number, the first subscript m taken modulo 6.

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Assuming a standard orientation of the given hexagon $\mathcal{H}_1$ in the complex plane, we easily determine the vertices of the hexagons in the above iterations.

If n is even, then $H_{1,n}H_{2,n}$, $H_{3,n}H_{4,n}$ and $H_{5,n}H_{6,n}$ are the opposite sides of the squares erected on $H_{1,n-1}H_{2,n-1}$, $H_{3,n-1}H_{4,n-1}$ and $H_{5,n-1}H_{6,n-1}$ respectively. This means, for $k = 1, 2, 3$,

$$\begin{aligned} H_{2k-1,n} =& H_{2k-1,n-1} - i(H_{2k,n-1} - H_{2k-1,n-1}) \\ =&(1+i)H_{2k-1,n-1} - i \cdot H_{2k,n-1}, \qquad (1) \\ H_{2k,n} =& H_{2k,n-1} + i(H_{2k-1,n-1} - H_{2k,n-1}) \\ =& i \cdot H_{2k-1,n-1} + (1-i)H_{2k,n-1}. \qquad (2) \end{aligned}$$

If n is odd, then $H_{2,n}H_{3,n}$, $H_{4,n}H_{5,n}$, $H_{6,n}H_{1,n}$ are the opposite sides of the squares erected on $H_{2,n-1}H_{3,n-1}$, $H_{4,n-1}H_{5,n-1}$ and $H_{6,n-1}H_{1,n-1}$ respectively. This means, for $k = 1, 2, 3$, reading first subscripts modulo 6, we have

$$\begin{aligned} H_{2k,n} =& H_{2k,n-1} - i(H_{2k+1,n-1} - H_{2k,n-1}) \\ =&(1+i)H_{2k,n-1} - i \cdot H_{2k+1,n-1}, \qquad (3) \\ H_{2k+1,n} =& H_{2k+1,n-1} + i(H_{2k,n-1} - H_{2k+1,n-1}) \\ =& i \cdot H_{2k,n-1} + (1-i)H_{2k+1,n-1}. \qquad (4) \end{aligned}$$

The above recurrence relations (1, 2, 3, 4) may be combined into

$$\begin{aligned} H_{2k,n} =&(1+(-1)^n i)H_{2k,n-1} + (-1)^n i \cdot H_{2k+(-1)^{n+1},n-1}, \\ H_{2k+1,n} =&(1+(-1)^n i)H_{2k+1,n-1} + (-1)^{n+1} i \cdot H_{2k+1+(-1)^n,n-1}, \end{aligned}$$

or even more succintly,

$$H_{m,n} = (1+(-1)^n i)H_{m,n-1} + (-1)^{m+n} i \cdot H_{m+(-1)^{m+n+1},n-1}.$$

Proposition 1. *Triangles $H_{1,n}H_{3,n}H_{5,n}$ and $H_{1,n-2}H_{3,n-2}H_{5,n-2}$ have the same centroid, so do triangles $H_{2,n}H_{4,n}H_{6,n}$ and $H_{2,n-2}H_{4,n-2}H_{6,n-2}$.*

Proof. Applying the relations (1, 2, 3, 4) twice, we have

$$\begin{aligned} H_{1,n} =& -(1-i)H_{6,n-2} + 2H_{1,n-2} + (1-i)H_{2,n-2} - H_{3,n-2}, \\ H_{3,n} =& -(1-i)H_{2,n-2} + 2H_{3,n-2} + (1-i)H_{4,n-2} - H_{5,n-2}, \\ H_{5,n} =& -(1-i)H_{4,n-2} + 2H_{5,n-2} + (1-i)H_{6,n-2} - H_{1,n-2}. \end{aligned}$$

The triangle $H_{1,n}H_{3,n}H_{5,n}$ has centroid

$$\frac{1}{3}\left(H_{1,n} + H_{3,n} + H_{5,n}\right) = \frac{1}{3}\left(H_{1,n-2} + H_{3,n-2} + H_{5,n-2}\right),$$

which is the centroid of triangle $H_{1,n-2}H_{3,n-2}H_{5,n-2}$. The proof for the other pair is similar. □

Theorem 2. *For each $m = 1, 2, 3, 4, 5, 6$, the sequence of vertices $H_{m,n}$ satisfies the recurrence relation*

$$H_{m,n} = 6H_{m,n-2} - 6H_{m,n-4} + H_{m,n-6}. \tag{5}$$

Proof. By using the recurrence relations (1, 2, 3, 4), we have

$$\begin{aligned}
H_{1,2} =&(1+i)H_{1,1} - iH_{2,1},\\
H_{1,3} =&2H_{1,1} - (1+i)H_{2,1} - H_{5,1} + (1+i)H_{6,1},\\
H_{1,4} =&3(1+i)H_{1,1} - 4iH_{2,1} - (1+i)H_{3,1} + iH_{4,1} - (1+i)H_{5,1} + 2iH_{6,1},\\
H_{1,5} =&8H_{1,1} - 5(1+i)H_{2,1} - H_{3,1} - 6H_{5,1} + 5(1+i)H_{6,1},\\
H_{1,6} =&13(1+i)H_{1,1} - 18iH_{2,1} - 6(1+i)H_{3,1} + 6iH_{4,1} - 6(1+i)H_{5,1} + 11iH_{6,1},\\
H_{1,7} =&37H_{1,1} - 24(1+i)H_{2,1} - 6H_{3,1} - 30H_{5,1} + 24(1+i)H_{6,1}.
\end{aligned}$$

Elimination of $H_{m,1}$, $m = 2, 3, 4, 5, 6$, from these equations gives

$$H_{1,7} = 6H_{1,5} - 6H_{1,3} + H_{1,1}.$$

The same relations hold if we simultaneously increase each first subscript by 2, or each second subscript by 1. Thus, we have the recurrence relation (5) for $m = 1, 3, 5$. Similarly,

$$\begin{aligned}
H_{2,2} =&iH_{1,1} + (1-i)H_{2,1},\\
H_{2,3} =&-(1-i)H_{1,1} + 2H_{2,1} + (1-i)H_{3,1} - H_{4,1},\\
H_{2,4} =&4iH_{1,1} + 3(1-i)H_{2,1} - 2iH_{3,1} - (1-i)H_{4,1} - iH_{5,1} - (1-i)H_{6,1},\\
H_{2,5} =&-5(1-i)H_{1,1} + 8H_{2,1} + 5(1-i)H_{3,1} - 6H_{4,1} - H_{6,1},\\
H_{2,6} =&18iH_{1,1} + 13(1-i)H_{2,1} - 11iH_{3,1} - 6(1-i)H_{4,1} - 6iH_{5,1} - 6(1-i)H_{6,1},\\
H_{2,7} =&-24(1-i)H_{1,1} + 37H_{2,1} + 24(1-i)H_{3,1} - 30H_{4,1} - 6H_{6,1}.
\end{aligned}$$

Elimination of $H_{m,1}$, $m = 1, 3, 4, 5, 6$, from these equations gives

$$H_{2,7} - 6H_{2,5} + 6H_{2,3} - H_{2,1} = 0.$$

A similar reasoning shows that (5) holds for $m = 2, 4, 6$. □

2. Midpoint triangles

Let M_1, M_2, M_3 be the midpoints of $H_{4,1}H_{5,1}$, $H_{6,1}H_{1,1}$ and $H_{2,1}H_{3,1}$ and M_1', M_2', M_3' the midpoints of $H_{1,3}H_{2,3}$, $H_{3,3}H_{4,3}$ and $H_{5,3}H_{6,3}$ respectively. We have

$$\begin{aligned}
M_1' =&\frac{1}{2}\left(H_{1,3} + H_{2,3}\right)\\
=&\frac{1}{2}\left((1+i)H_{1,1} + (1-i)H_{2,1} + (1-i)H_{3,1} - H_{4,1} - H_{5,1} + (1+i)H_{6,1}\right)\\
=&-M_1 + (1+i)M_2 + (1-i)M_3.
\end{aligned}$$

Similarly,

$$M_2' = (1-i)M_1 - M_2 + (1+i)M_3,$$
$$M_3' = (1+i)M_1 + (1-i)M_2 - M_3.$$

Proposition 3. *For a permutation (j,k,ℓ) of the integers $1,2,3$, the segments M_jM_k' and M_kM_j' are perpendicular to each other and equal in length, while $M_\ell M_\ell'$ is parallel to an angle bisector of M_jM_k' and M_kM_j', and is $\sqrt{2}$ times as long as each of these segments.*

Proof. From the above expressions for M_j', $j=1,2,3$, we have

$$M_2' - M_3 = (1-i)M_1 - M_2 + i\cdot M_3, \tag{6}$$
$$\begin{aligned} M_3' - M_2 &= (1+i)M_1 - i\cdot M_2 - M_3,\\ &= i\left((1-i)M_1 - M_2 + iM_3\right),\\ &= i(M_2' - M_3); \end{aligned} \tag{7}$$
$$\begin{aligned} M_1 - M_1' &= 2M_1 - (1+i)M_2 - (1-i)M_3\\ &= (M_2' - M_3) + (M_3' - M_2). \end{aligned} \tag{8}$$

From (6) and (7), M_2M_3' and M_3M_2' are perpendicular and have equal lengths. From (8), we conclude that $M_1'M_1$ is parallel to an angle bisector of M_2M_3' and M_3M_2', and is $\sqrt{2}$ times as long as each of these segments. The same results for $(k,\ell) = (3,1),\ (1,2)$ follow similarly. □

The midpoints of the segments M_jM_j', $j=1,2,3$, are the points

$$N_1 = \frac{1}{2}\left((1+i)M_2 + (1-i)M_3\right),$$
$$N_2 = \frac{1}{2}\left((1+i)M_3 + (1-i)M_1\right),$$
$$N_3 = \frac{1}{2}\left((1+i)M_1 + (1-i)M_2\right).$$

Note that

$$\begin{aligned} N_1 &= \frac{M_2+M_3}{2} + i\cdot\frac{M_2-M_3}{2},\\ &= \frac{M_2'+M_3'}{2} - i\cdot\frac{M_2'-M_3'}{2}. \end{aligned}$$

Thus, N_1 is the center of the square constructed externally on the side M_2M_3 of triangle $M_1M_2M_3$, and also the center of the square constructed internally on $M_1'M_2'M_3'$. Similarly, for N_2 and N_3. From this we deduce the following corollaries. See Figure 2.

Corollary 4. *The triangles $M_1M_2M_3$ and $M_1'M_2'M_3'$ are perspective. The perspector is the outer Vecten point of $M_1M_2M_3$ and the inner Vecten point of $M_1'M_2'M_3'$.* [1]

Corollary 5. *The segments M_jN_ℓ and M_kN_ℓ are equal in length and are perpendicular. The same is true for $M_j'N_\ell$ and $M_k'N_\ell$.*

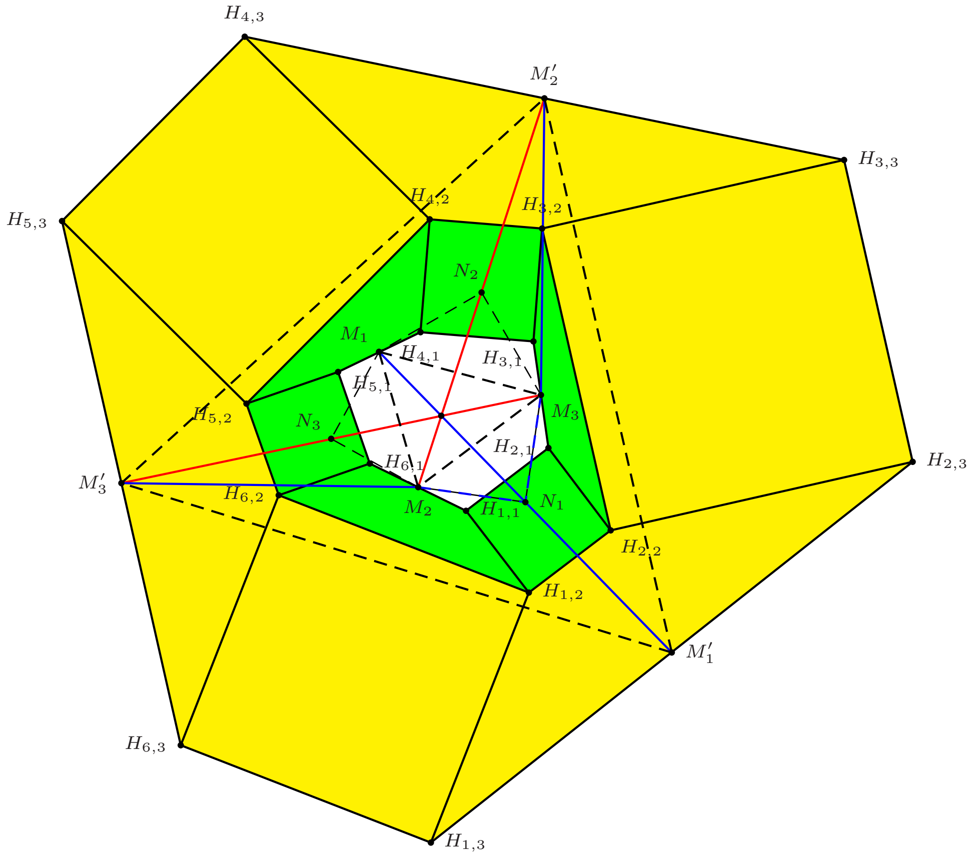

Figure 2

Let $M_1''M_2''M_3''$ be the desmic mate of $M_1M_2M_3$ and $M_1'M_2'M_3'$, *i.e.*, $M_1'' = M_2M_3' \cap M_3M_2'$ etc. By Proposition 3, $\angle M_2M_1''M_3$ is a right angle, and M_1'' lies on the circle with diameter M_2M_3. Since the bisector angle $M_2M_1''M_3$ is parallel to the line N_1M_1, $M_1''N_1$ is perpendicular to this latter line. See Figure 3.

Proposition 6. *If (j, k, ℓ) is a permutation of 1, 2, 3, the lines M_jM_k' and M_kM_j' and the line through N_ℓ perpendicular to $M_\ell M_\ell'$ are concurrent (at M_ℓ'').*

Corollary 7. *The circles $(M_jM_kN_\ell)$, $(M_j'M_k'N_\ell)$, $(M_\ell M_\ell''N_\ell)$ and $(M_\ell'M_\ell''N_\ell)$ are coaxial, so the midpoints of M_jM_k, $M_j'M_k'$, $M_\ell M_\ell''$ and $M_\ell'M_\ell''$ are collinear, the line being parallel to $M_\ell M_\ell'$.*

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[1] The outer (respectively inner) Vecten point is the point X_{485} (respectively X_{486}) of [4].

Since the lines $M_jM'_j$, $j = 1, 2, 3$, concur at the outer Vecten point of triangle $M_1M_2M_3$, the intersection of the lines is the inferior (complement) of the outer Vecten point. [2] As such, it is the center of the circle through $N_1N_2N_3$ (see [4]).

Corollary 8. *The three lines joining the midpoints of $M_1M'_1$, $M_2M'_2$, $M_3M'_3$ are concurrent at the center of the circle through N_1, N_2, N_3, which also passes through M''_1, M'_2 and M''_3.*

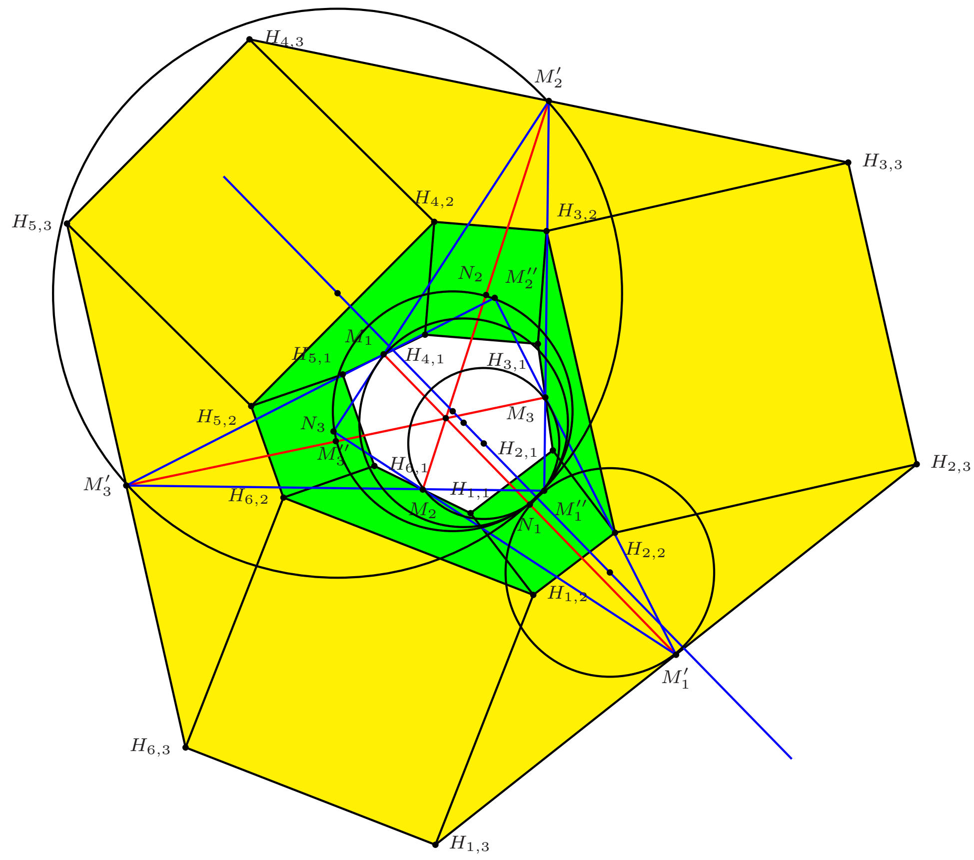

Figure 3

3. Starting with a triangle

An interesting special case occurs when the initial hexagon $\mathcal{H}_1$ degenerates into a triangle with

$$H_{1,1} = H_{6,1} = A, \qquad H_{2,1} = H_{3,1} = B, \qquad H_{4,1} = H_{5,1} = C.$$

This case has been studied before by Haight and Nottrot, who examined especially the side lengths and areas of the squares in each wreath. Under this assumption,

[2] X_{641} in [4].

between two consecutive hexagons $\mathcal{H}_n$ and $\mathcal{H}_{n+1}$ are three squares and three alternating trapezoids of equal areas. The trapzoids between $\mathcal{H}_1$ and $\mathcal{H}_2$ degenerate into triangles. The sides of the squares are parallel and perpendicular to the sides or to the medians of triangle ABC according as n is odd or even. We shall assume the sidelengths of triangle ABC to be a, b, c, and the median lengths m_a, m_b, m_c respectively.

The squares of the first wreath are attached to the triangle sides outwardly. [3] Haight [2] has computed the ratios of the sidelengths of the squares.

If $n = 2k - 1$, the squares have sidelengths a, b, c multiplied by $a_1(k)$, where

$$a_1(k) = 5a_1(k-1) - a_1(k-2),$$
$$a_1(1) = 1,\ a_1(2) = 4.$$

This is sequence A004253 in Sloane's *Online Encyclopedia of Integer sequences* [9]. This also means that

$$H_{m,2k} - H_{m,2k-1} = a_1(k)(H_{m,2} - H_{m,1}). \tag{9}$$

If $n = 2k$, the squares have sidelengths $2m_a$, $2m_b$, $2m_c$ multiplied by $a_2(k)$, where

$$a_2(k) = 5a_2(k-1) - a_2(k-2),$$
$$a_2(1) = 1,\ a_2(2) = 5.$$

This is sequence A004254 in [9]. This also means that

$$H_{m,2k+1} - H_{m,2k} = a_2(k)(H_{m,3} - H_{m,2}). \tag{10}$$

Proposition 9. *Each trapezoid in the wreath bordered by $\mathcal{H}_n$ and $\mathcal{H}_{n+1}$ has area $a_2(n) \cdot \triangle ABC$.*

Lemma 10. (1) $\sum_{j=1}^{k} a_1(j) = a_2(k)$.
(2) *The sums $a_3(k) = \sum_{j=1}^{k} a_2(j)$ satisfy the recurrence relation*

$$a_3(k) = 6a_3(k-1) - 6a_3(k-2) + a_3(k-3),$$
$$a_3(1) = 1,\ a_3(2) = 6,\ a_3(3) = 30.$$

The sequence $a_3(k)$ is essentially sequence A089817 in [9]. [4]

It follows from (9) and (10) that

$$\begin{aligned} H_{m,2k} =& H_{m,1} + \sum_{j=1}^{k}(H_{m,2j} - H_{m,2j-1}) + \sum_{j=1}^{k-1}(H_{m,2j+1} - H_{m,2j}) \\ =& H_{m,1} + \left(\sum_{j=1}^{k} a_1(j)\right)(H_{m,2} - H_{m,1}) + \left(\sum_{j=1}^{k-1} a_2(j)\right)(H_{m,3} - H_{m,2}) \\ =& H_{m,1} + a_2(k)(H_{m,2} - H_{m,1}) + a_3(k-1)(H_{m,3} - H_{m,2}). \end{aligned}$$

[3] Similar results as those in §§3, 4 can be found if these initial squares are constructed inwardly.

[4] Note that sequences a_1 and a_2 follow this third order recurrence relation as well.

Also,

$$\begin{aligned}H_{m,2k+1} =&H_{m,1}+\sum_{j=1}^{k}(H_{m,2j}-H_{m,2j-1})+\sum_{j=1}^{k}(H_{m,2j+1}-H_{m,2j})\\ =&H_{m,1}+\left(\sum_{j=1}^{k}a_1(j)\right)(H_{m,2}-H_{m,1})+\left(\sum_{j=1}^{k}a_2(j)\right)(H_{m,3}-H_{m,2})\\ =&H_{m,1}+a_2(k)(H_{m,2}-H_{m,1})+a_3(k)(H_{m,3}-H_{m,2})\end{aligned}$$

Here is a table of the *absolute* barycentric coordinates (with respect to triangle ABC) of the initial values in the above recurrence relations.

m	$H_{m,1}$	$H_{m,2}-H_{m,1}$	$H_{m,3}-H_{m,2}$
1	$(1,0,0)$	$\frac{1}{S}(S_B,S_A,-c^2)$	$(2,-1,-1)$
2	$(0,1,0)$	$\frac{1}{S}(S_B,S_A,-c^2)$	$(-1,2,-1)$
3	$(0,1,0)$	$\frac{1}{S}(-a^2,S_C,S_B)$	$(-1,2,-1)$
4	$(0,0,1)$	$\frac{1}{S}(-a^2,S_C,S_B)$	$(-1,-1,2)$
5	$(0,0,1)$	$\frac{1}{S}(S_C,-b^2,S_A)$	$(-1,-1,2)$
6	$(1,0,0)$	$\frac{1}{S}(S_C,-b^2,S_A)$	$(2,-1,-1)$

From these we have the homogeneous barycentric coordinates

$$\begin{aligned}H_{m,2k} =&H_{m,1}+a_2(k)(H_{m,2}-H_{m,1})+a_3(k-1)(H_{m,3}-H_{m,2}),\\ H_{m,2k+1} =&H_{m,1}+a_2(k)(H_{m,2}-H_{m,1})+a_3(k)(H_{m,3}-H_{m,2}).\end{aligned}$$

These can be combined into a single relation

$$H_{m,n}=H_{m,1}+a_2(n')(H_{m,2}-H_{m,1})+a_3(n'')(H_{m,3}-H_{m,2}),$$

in which $n'=\lfloor\frac{n}{2}\rfloor$ and $n''=\lfloor\frac{n-1}{2}\rfloor$.

Here are the coordinates of the points $H_{m,n}$.

m	x	y	z
1	$(2a_3(n'')+1)S+a_2(n')S_B$	$-a_3(n'')S+a_2(n')S_A$	$-a_3(n'')S-a_2(n')c^2$
2	$-a_3(n'')S+a_2(n')S_B$	$(2a_3(n'')+1)S+a_2(n')S_A$	$-a_3(n'')S-a_2(n')c^2$
3	$-a_3(n'')S-a_2(n')a^2$	$(2a_3(n'')+1)S+a_2(n')S_C$	$-a_3(n'')S+a_2(n')S_B$
4	$-a_3(n'')S-a_2(n')a^2$	$-a_3(n'')S+a_2(n')S_C$	$(2a_3(n'')+1)S+a_2(n')S_B$
5	$-a_3(n'')S+a_2(n')S_C$	$-a_3(n'')S-a_2(n')b^2$	$(2a_3(n'')+1)S+a_2(n')S_A$
6	$(2a_3(n'')+1)S+a_2(n')S_C$	$-a_3(n'')S-a_2(n')b^2$	$-a_3(n'')S+a_2(n')S_A$

Note that the coordinate sum of each of the points in the above is equal to S.

Consider the midpoints of the following segments

segment	$H_{1,n}H_{2,n}$	$H_{2,n}H_{3,n}$	$H_{3,n}H_{4,n}$	$H_{4,n}H_{5,n}$	$H_{5,n}H_{6,n}$	$H_{6,n}H_{1,n}$
midpoint	$C_{1,n}$	$B_{2,n}$	$A_{1,n}$	$C_{2,n}$	$B_{1,n}$	$A_{2,n}$

For $j = 1, 2$, denote by $\mathcal{T}_{j,n}$ the triangle with vertices $A_{j,n}B_{j,n}C_{j,n}$.

$A_{1,n}$	$-2a_3(n'')S - 2a_2(n')a^2$	$(a_3(n'')+1)S + 2a_2(n')S_C$	$(a_3(n'')+1)S + 2a_2(n')S_B$
$B_{1,n}$	$(a_3(n'')+1)S + 2a_2(n')S_C$	$-2a_3(n'')S - 2a_2(n')b^2$	$(a_3(n'')+1)S + 2a_2(n')S_A$
$C_{1,n}$	$(a_3(n'')+1)S + 2a_2(n')S_B$	$(a_3(n'')+1)S + 2a_2(n')S_A$	$-2a_3(n'')S - 2a_2(n')c^2$
$A_{2,n}$	$-2(2a_3(n'')+1)S - a_2(n')a^2$	$2a_3(n'')S + a_2(n')S_C$	$2a_3(n'')S + a_2(n')S_B$
$B_{2,n}$	$2a_3(n'')S + a_2(n')S_C$	$-2(2a_3(n'')+1)S - a_2(n')b^2$	$2a_3(n'')S + a_2(n')S_A$
$C_{2,n}$	$2a_3(n'')S + a_2(n')S_B$	$2a_3(n'')S + a_2(n')S_A$	$-2(2a_3(n'')+1)S - a_2(n')c^2$

Proposition 11. *The triangles $\mathcal{T}_{1,n}$ and $\mathcal{T}_{2,n}$ are perspective.*

This is a special case of the following general result.

Theorem 12. *Every triangle of the form*

$$\begin{array}{lclcl} -2fS - ga^2 & : & (f+1)S + gS_C & : & (f+1)S + gS_B \\ (f+1)S + gS_C & : & -2fS - gb^2 & : & (f+1)S + gS_A \\ (f+1)S + gS_B & : & (f+1)S + gS_A & : & -2fS - gc^2 \end{array}$$

where f and g represent real numbers, is perspective with the reference triangle. Any two such triangles are perspective.

Proof. Clearly the triangle given above is perspective with ABC at the point

$$\left(\frac{1}{(f+1)S + gS_A} : \frac{1}{(f+1)S + gS_B} : \frac{1}{(f+1)S + gS_C}\right),$$

which is the Kiepert perspector $K(\phi)$ for $\phi = \cot^{-1}\frac{f+1}{g}$.

Consider a second triangle of the same form, with f and g replaced by p and q respectively. We simply give a description of this perspector P. This perspector is the centroid if and only if $(g-q) + 3(gp - fq) = 0$. Otherwise, the line joining this perspector to the centroid G intersects the Brocard axis at the point

$$Q = (a^2((g-q)S_A - (f-p)S) : b^2((g-q)S_B - (f-p)S) : c^2((g-q)S_C - (f-p)S)),$$

which is the isogonal conjugate of the Kiepert perspector $K\left(-\cot^{-1}\frac{f-p}{g-q}\right)$. The perspector P in question divides GQ in the ratio

$$\begin{aligned} & GP : GQ \\ =& ((g-q) + 3(gp - fq))((g-q)S - (f-p)S_\omega) \\ & : (3(f-p)^2 + (g-q)^2)S + 2(f-p)(g-q)S_\omega. \end{aligned}$$

□

Note that $\mathcal{T}_{j,n}$ for $j \in \{1, 2\}$ and the Kiepert triangles $\mathcal{K}_\phi$[5] are of this form. Also the medial triangle of a triangle of this form is again of the same form. The perspectors of $\mathcal{T}_{j,n}$ and ABC lie on the Kiepert hyperbola. It is the Kiepert perspector $K(\phi_{j,n})$ where

$$\cot \phi_{1,n} = \frac{a_3(n'')+1}{2a_2(n')},$$

[5] See for instance [6].

and

$$\cot \phi_{2,n} = \frac{2a_3(n'')}{a_2(n')}.$$

In particular the perspectors tend to limits when n tends to infinity. The perspectors and limits are given by

triangle	perspector K_ϕ with	limit for $k \to \infty$
$\mathcal{T}_{1,2k}$	$\phi_{1,2k} = \cot^{-1} \frac{a_3(k-1)+1}{2a_2(k)}$	$\phi_{1,\text{even}} = \cot^{-1} \frac{\sqrt{21}-3}{12}$
$\mathcal{T}_{1,2k+1}$	$\phi_{1,2k+1} = \cot^{-1} \frac{a_3(k)+1}{2a_2(k)}$	$\phi_{1,\text{odd}} = \cot^{-1} \frac{\sqrt{21}+3}{12}$
$\mathcal{T}_{2,2k}$	$\phi_{2,2k} = \cot^{-1} \frac{2a_3(k-1)}{a_2(k)}$	$\phi_{2,\text{even}} = \cot^{-1} \frac{\sqrt{21}-3}{3}$
$\mathcal{T}_{2,2k+1}$	$\phi_{2,2k+1} = \cot^{-1} \frac{2a_3(k)}{a_2(k)}$	$\phi_{2,\text{odd}} = \cot^{-1} \frac{\sqrt{21}+3}{3}$

Remarks. (1) $\mathcal{T}_{2,2}$ is the medial triangle of $\mathcal{T}_{1,3}$.

(2) The perspector of $\mathcal{T}_{2,2}$ and $\mathcal{T}_{2,3}$ is X_{591}.

(3) The perspector of $\mathcal{T}_{2,2}$ and $\mathcal{T}_{2,4}$ is the common circumcenter of $\mathcal{T}_{1,3}$ and $\mathcal{T}_{2,4}$.

Nottrot [8], on the other hand, has found that the sum of the areas of the squares between $\mathcal{H}_n$ and $\mathcal{H}_{n+1}$ as $a_4(n)(a^2+b^2+c^2)$, where wreaths.

$$a_4(n) = 4a_4(n-1) + 4a_4(n-2) - a_4(n-3),$$
$$a_4(1) = 1,\ a_4(2) = 3,\ a_4(3) = 16.$$

This is sequence A005386 in [9]. Note that $a_1(n) = a_4(n) + a_4(n-1)$ for $n \geq 2$.

4. Pairs of congruent triangles

Lemma 13. (a) *If $n \geq 2$ is even, then*

$$\overrightarrow{A_{1,n}A_{1,n+1}} = -\frac{1}{2}\overrightarrow{H_{6,n}H_{6,n+1}} = -\frac{1}{2}\overrightarrow{H_{1,n}H_{1,n+1}},$$
$$\overrightarrow{B_{1,n}B_{1,n+1}} = -\frac{1}{2}\overrightarrow{H_{2,n}H_{2,n+1}} = -\frac{1}{2}\overrightarrow{H_{3,n}H_{3,n+1}},$$
$$\overrightarrow{C_{1,n}C_{1,n+1}} = -\frac{1}{2}\overrightarrow{H_{4,n}H_{4,n+1}} = -\frac{1}{2}\overrightarrow{H_{5,n}H_{5,n+1}}.$$

(b) *If $n \geq 3$ is odd, then*

$$\overrightarrow{A_{2,n}A_{2,n+1}} = -\frac{1}{2}\overrightarrow{H_{3,n}H_{3,n+1}} = -\frac{1}{2}\overrightarrow{H_{4,n}H_{4,n+1}},$$
$$\overrightarrow{B_{2,n}B_{2,n+1}} = -\frac{1}{2}\overrightarrow{H_{5,n}H_{5,n+1}} = -\frac{1}{2}\overrightarrow{H_{6,n}H_{6,n+1}},$$
$$\overrightarrow{C_{2,n}C_{2,n+1}} = -\frac{1}{2}\overrightarrow{H_{1,n}H_{1,n+1}} = -\frac{1}{2}\overrightarrow{H_{2,n}H_{2,n+1}}.$$

Proof. Consider the case of $A_{1,n}A_{1,n+1}$ for even n. Translate the trapezoid $H_{5,n+1}H_{6,n+1}H_{6,n}H_{5,n}$ by the vector $\overrightarrow{H_{5,n+1}H_{4,n+1}}$ and the trapezoid $H_{2,n+1}H_{2,n}H_{1,n}H_{1,n+1}$ by the vector $\overrightarrow{H_{2,n+1}H_{3,n+1}}$. Together with the trapezoid $H_{3,n}H_{4,n}H_{4,n+1}H_{3,n+1}$, these images form two triangles $XH_{3,n+1}H_{4,n+1}$ and

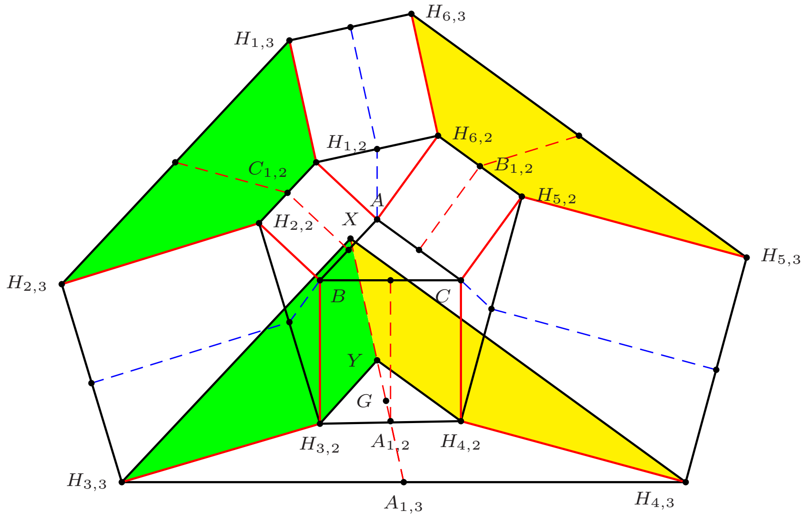

Figure 4.

$YH_{3,n}H_{4,n}$ homothetic at their common centroid G. [6] See Figure 4. It is clear that the points X, Y, $A_{1,n}$, $A_{1,n+1}$ all lie on a line through the centroid G. Furthermore, $\overrightarrow{A_{1,n}A_{1,n+1}} = \frac{1}{2}\overrightarrow{XY} = -\frac{1}{2}\overrightarrow{H_{1,n}H_{1,n+1}}$. The other cases follow similarly. □

Proposition 14. (1) *If $n \geq 3$ is odd, the following pairs of triangles are congruent.*
(i) $H_{2,n}B_{2,n+1}C_{1,n-1}$ *and* $H_{5,n}B_{1,n-1}C_{2,n+1}$,
(ii) $H_{3,n}A_{1,n-1}B_{2,n+1}$ *and* $H_{6,n}A_{2,n+1}B_{1,n-1}$,
(iii) $H_{4,n}C_{2,n+1}A_{1,n-1}$ *and* $H_{1,n}C_{1,n-1}A_{2,n+1}$.

(2) *If $n \geq 2$ is even, the following pairs of triangles are congruent.*
(iv) $H_{2,n}B_{2,n-1}C_{1,n+1}$ *and* $H_{5,n}B_{1,n+1}C_{2,n-1}$,
(v) $H_{3,n}A_{1,n+1}B_{2,n-1}$ *and* $H_{6,n}A_{2,n-1}B_{1,n+1}$,
(vi) $H_{4,n}C_{2,n-1}A_{1,n+1}$ *and* $H_{1,n}C_{1,n+1}A_{2,n-1}$.

Proof. We consider the first of these cases. Let $n \geq 3$ be an odd number. Consider the triangles $H_{2,n}B_{2,n+1}C_{1,n-1}$ and $H_{5,n}B_{1,n-1}C_{2,n+1}$. We show that $H_{2,n}B_{2,n+1}$ and $H_{5,n}B_{1,n-1}$ are perpendicular to each other and equal in length, and the same for $H_{2,n}C_{1,n-1}$ and $H_{5,n}C_{2,n+1}$.

Consider the triangles $H_{2,n}B_{2,n}B_{2,n+1}$ and $B_{1,n-1}B_{1,n}H_{5,n}$. By Lemma 13, $B_{2,n}B_{2,n+1}$ is parallel to $H_{5,n}H_{5,n+1}$ and is half its length. It follows that $B_{2,n}B_{2,n+1}$ is perpendicular to and has the same length as $B_{1,n}H_{5,n}$. Similarly, $B_{2,n}H_{2,n}$ is perpendicular to and has the same length as $B_{1,n}B_{1,n-1}$. Therefore, the triangles $H_{2,n}B_{2,n}B_{2,n+1}$ and $B_{1,n-1}B_{1,n}H_{5,n}$ are congruent, and the segments $H_{2,n}B_{2,n+1}$

[6] As noted in the beginning of §3, the sides of the squares are parallel to and perpendicular to the sides of ABC or the medians of ABC according as n is odd or even. In the even case it is thus clear that the homothetic center is the centroid. In the odd case it can be seen as the lines perpendicular to the sides of ABC are parallel to the medians of the triangles in the first wreath (the flank triangles). The centroid and the orthocenter befriend each other. See [5].

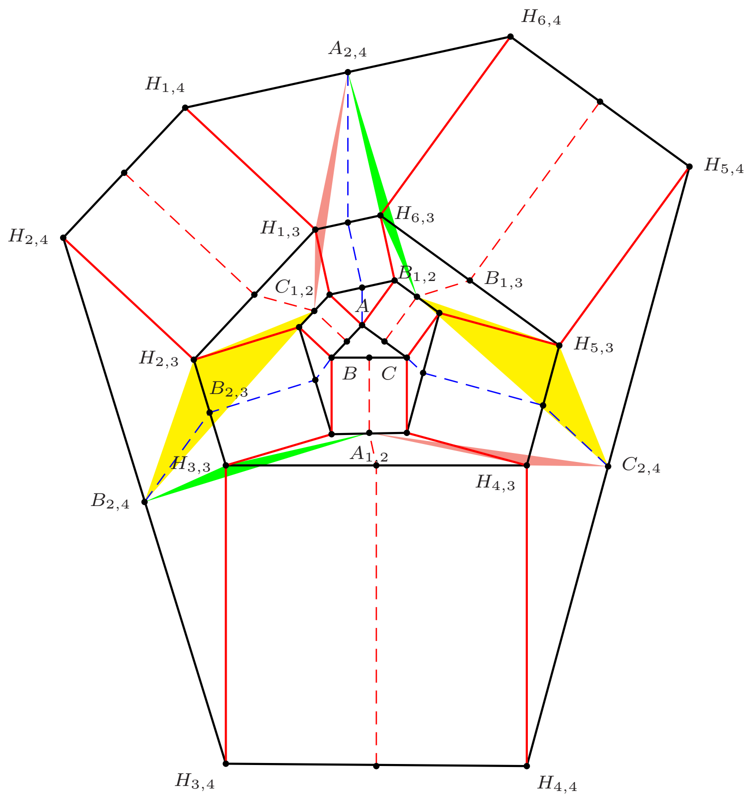

Figure 5.

and $B_{1,n-1}H_{5,n}$ are perpendicular and equal in length. See Figure 5. The same reasoning shows that the segments $H_{2,n}C_{1,n-1}$ and $H_{5,n}C_{2,n+1}$ are perpendicular and equal in length. Therefore, the triangles $H_{2,n}B_{2,n+1}C_{1,n-1}$ and $H_{5,n}B_{1,n-1}C_{2,n+1}$ are congruent.

The other cases can be proved similarly. □

Remark. The fact that the segments $B_{2,n+1}C_{1,n-1}$ and $B_{1,n-1}C_{2,n+1}$ are perpendicular and equal in length has been proved in Proposition 3 for square wreaths arising from an arbitrary hexagon.

5. A pair of Kiepert hyperbolas

The triangles $A_{1,3}H_{4,2}H_{3,2}$, $H_{5,2}B_{1,3}H_{6,2}$ and $H_{2,2}H_{1,2}C_{1,3}$ are congruent to ABC. The counterparts of a point P in these triangles are the points with the barycentric coordinates relative to these three triangles as P relative to ABC.

Theorem 15. *The locus of point P in ABC whose counterparts in the triangles $A_{1,3}H_{4,2}H_{3,2}$, $H_{5,2}B_{1,3}H_{6,2}$ and $H_{2,2}H_{1,2}C_{1,3}$ form a triangle $A'B'C'$ perspective to ABC is the union of the line at infinity and the rectangular hyperbola*

$$S\sum_{\text{cyclic}}(S_B-S_C)yz+(x+y+z)\left(\sum_{\text{cyclic}}(S_B-S_C)(S_A+S)x\right)=0. \quad (11)$$

The locus of the perspector is the union of the line at infinity and the Kiepert hyperbola of triangle ABC.

Proof. The counterparts of $P=(x:y:z)$ form a triangle perspective with ABC if and only if the parallels through A, B, C to $A_{1,3}P$, $B_{1,3}P$ and $C_{1,3}P$ are concurrent. These parallels have equations

$$\begin{aligned}
((S+S_B)(x+y+z)-Sz)Y-((S+S_C)(x+y+z)-Sy)Z&=0,\\
-((S+S_A)(x+y+z)-Sz)X+((S+S_C)(x+y+z)-Sx)Z&=0,\\
((S+S_A)(x+y+z)-Sy)X-((S+S_B)(x+y+z)-Sx)Y&=0.
\end{aligned}$$

They are concurrent if and only if

$$(x+y+z)\left(\sum_{cyclic}(b^2-c^2)((S_A+S)x^2-S_Ayz)\right)=0.$$

The locus therefore consists of the line at infinity and a conic. Rearranging the equation of the conic in the form (11), we see that it is homothetic to the Kiepert hyperbola.

For a point P on the locus (11), let $Q=(x:y:z)$ be the corresponding perspector. This means that the parallels through $A_{1,3}$, $B_{1,3}$, $C_{1,3}$ to AQ, BQ, CQ are concurrent. These parallels have equations

$$\begin{aligned}
((S+S_B)y-(S+S_C)z)X+((S+S_B)y-S_Cz)Y-((S+S_C)z-S_By)Z&=0,\\
-((S+S_A)x-S_Cz)X+((S+S_C)z-(S+S_A)x)Y+((S+S_C)z-S_Ax)Z&=0,\\
((S+S_A)x-S_By)X-((S+S_B)y-S_Ax)Y+((S+S_A)x-(S+S_B)y)Z&=0.
\end{aligned}$$

They are concurrent if and only if

$$(x+y+z)((S_B-S_C)yz+(S_C-S_A)zx+(S_A-S_B)xy)=0.$$

This means the perspector lies on the union of the line at infinity and the Kiepert hyperbola. □

Here are some examples of points on the locus (11) with the corresponding perspectors on the Kiepert hyperbola (see [1, 7]).

Q on Kiepert hyperbola	P on locus
$K\left(\arctan\frac{3}{2}\right)$	centroid
orthocenter	de Longchamps point
centroid	$G' = (-2S_A + a^2 + S : \cdots : \cdots)$
outer Vecten point	outer Vecten point
A	$A' = B_{1,3}H_{5,2} \cap C_{1,3}H_{2,2}$
B	$B' = C_{1,3}H_{1,2} \cap A_{1,3}H_{4,2}$
C	$C' = A_{1,3}H_{3,2} \cap B_{1,3}H_{6,2}$
A_0	$A_{1,3} = (-(S + S_B + S_C) : S + S_C : S + S_B)$
B_0	$B_{1,3} = (S + S_C : -(S + S_C + S_A) : S + S_A)$
C_0	$C_{1,3} = (S + S_B : S + S_A : -(S + S_A + S_B))$

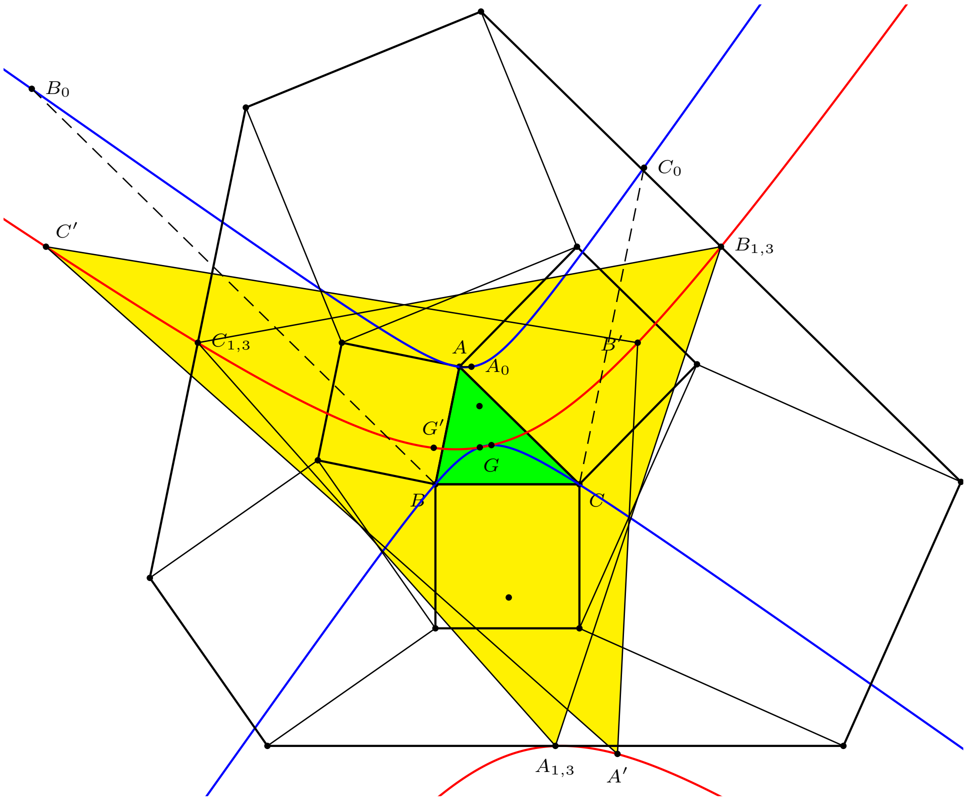

Figure 6.

Here, A_0 is the intersection of the Kiepert hyperbola with the parallel through A to BC; similarly for B_0 and C_0. Since the triangle $A_{1,3}B_{1,3}C_{1,3}$ has centroid G, the rectangular hyperbola (11) is the Kiepert hyperbola of triangle $A_{1,3}B_{1,3}C_{1,3}$. See Figure 6. We show that it is also the Kiepert hyperbola of triangle $A'B'C'$.

This follows from the fact that A', B', C' have coordinates

$$\begin{array}{llclcl} A' & -S-2S_A & : & S+S_A & : & S+S_A \\ B' & S+S_B & : & -S-2S_B & : & S+S_B \\ C' & S+S_C & : & S+S_C & : & -S-2S_C \end{array}$$

From these, the centroid of triangle $A'B'C'$ is the point G' in the above table. The rectangular hyperbola (11) is therefore the Kiepert hyperbola of triangle $A'B'C'$.

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