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Hansen's Right Triangle Theorem, Its Converse and a Generalization

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Abstract. We generalize D. W. Hansen's theorem relating the inradius and exradii of a right triangle and its sides to an arbitrary triangle. Specifically, given a triangle, we find two quadruples of segments with equal sums and equal sums of squares. A strong converse of Hansen's theorem is also established.

1. Hansen's right triangle theorem

In an interesting article in *Mathematics Teacher*, D. W. Hansen [2] has found some remarkable identities associated with a right triangle. Let ABC be a triangle with a right angle at C, sidelengths a, b, c. It has an incircle of radius r, and three excircles of radii r_a, r_b, r_c.

Theorem 1 (Hansen)**.** (1) *The sum of the four radii is equal to the perimeter of the triangle:*

$$r_a + r_b + r_c + r = a + b + c.$$

(2) *The sum of the squares of the four radii is equal to the sum of the squares of the sides of the triangle:*

$$r_a^2 + r_b^2 + r_c^2 + r^2 = a^2 + b^2 + c^2.$$

We seek to generalize Hansen's theorem to an arbitrary triangle, by replacing a, b, c by appropriate quantities whose sum and sum of squares are respectively equal to those of r_a, r_b, r_c and r. Now, for a right triangle ABC with right angle vertex C, this latter vertex is the orthocenter of the triangle, which we generically denote by H. Note that

$$a = BH \quad \text{and} \quad b = AH.$$

On the other hand, the hypotenuse being a diameter of the circumcircle, $c = 2R$. Note also that $CH = 0$ since C and H coincide. This suggests that a possible generalization of Hansen's theorem is to replace the triple a, b, c by the quadruple AH, BH, CH and $2R$. Since $AH = 2R\cos A$ etc., one of the quantities AH, BH, CH is negative if the triangle contains an obtuse angle.

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We shall establish the following theorem.

Theorem 2. *Let ABC be a triangle with orthocenter H and circumradius R.*
(1) $r_a + r_b + r_c + r = AH + BH + CH + 2R$;
(2) $r_a^2 + r_b^2 + r_c^2 + r^2 = AH^2 + BH^2 + CH^2 + (2R)^2$.

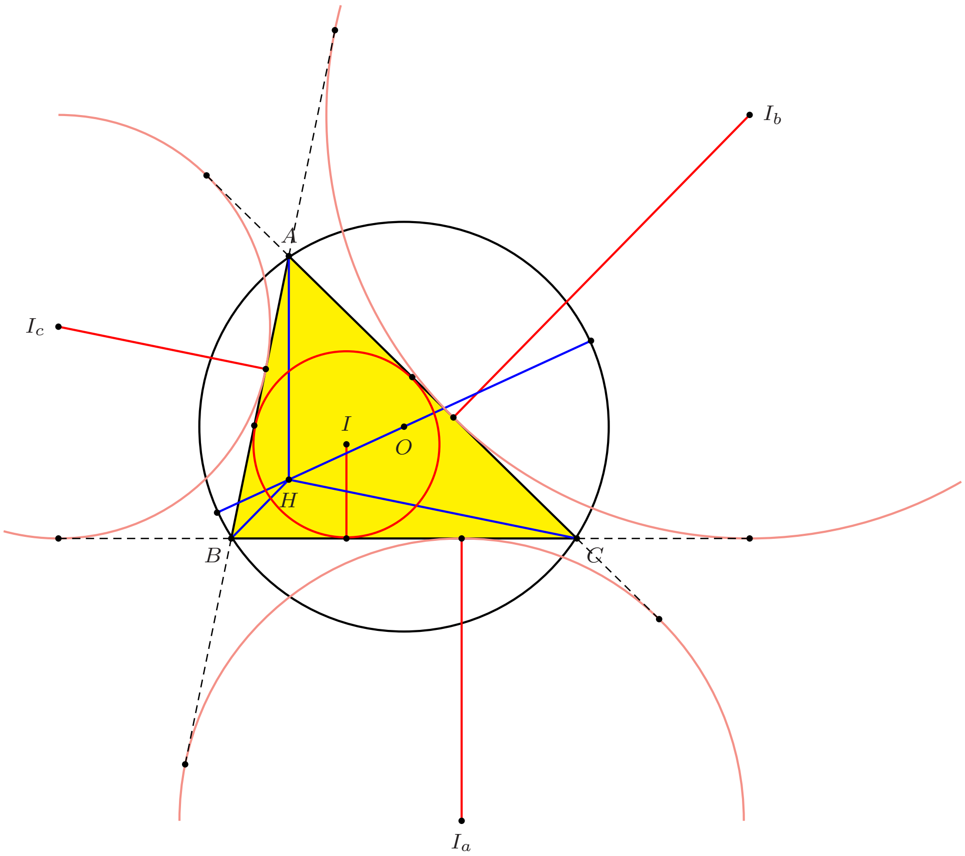

Figure 1. Two quadruples with equal sums and equal sums of squares

2. A characterization of right triangles in terms of inradius and exradii

Proposition 3. *The following statements for a triangle ABC are equivalent.*
(1) $r_c = s$.
(2) $r_a = s - b$.
(3) $r_b = s - a$.
(4) $r = s - c$.
(5) *C is a right angle.*

Proof. By the formulas for the exradii and the Heron formula, each of (1), (2), (3), (4) is equivalent to the condition

$$(s-a)(s-b) = s(s-c). \tag{1}$$

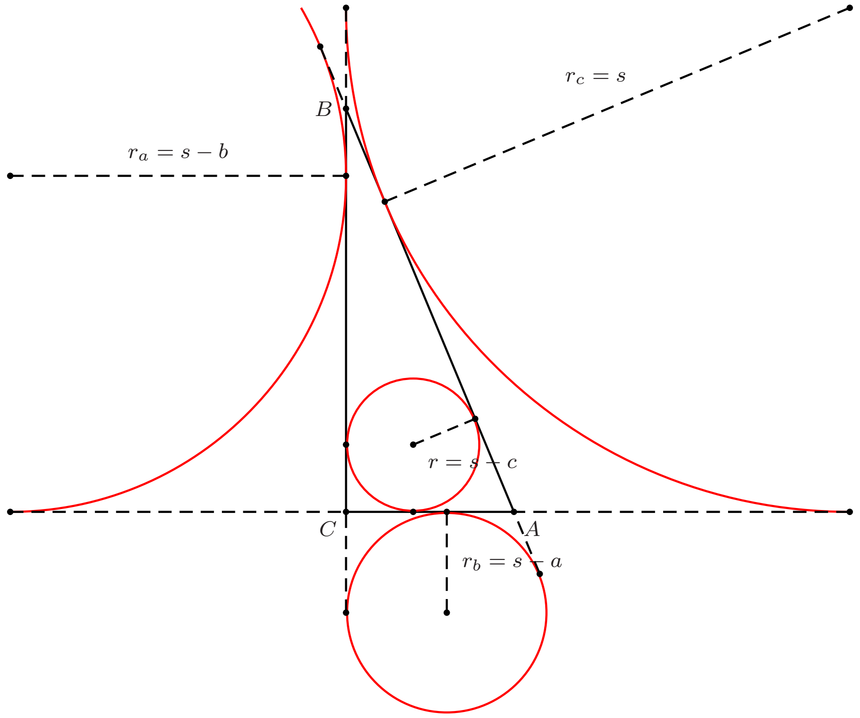

Figure 2. Inradius and exradii of a right triangle

Assuming (1), we have $s^2 - (a+b)s + ab = s^2 - cs$, $(a+b-c)s = ab$, $(a+b-c)(a+b+c) = 2ab$, $(a+b)^2 - c^2 = 2ab$, $a^2 + b^2 = c^2$. This shows that each of (1), (2), (3), (4) implies (5). The converse is clear. See Figure 2. □

3. A formula relating the radii of the various circles

As a preparation for the proof of Theorem 2, we study the excircles in relation to the circumcircle and the incircle. We establish a basic result, Proposition 6, below. Lemma 4 and the statement of Proposition 6 can be found in [3, pp.185–193]. An outline proof of Proposition 5 can be found in [4, §2.4.1]. Propositions 5 and 6 can also be found in [5, §4.6.1]. [1] We present a unified detailed proof of these propositions here, simpler and more geometric than the trigonometric proofs outlined in [3].

Consider triangle ABC with its circumcircle (O). Let the bisector of angle A intersect the circumcircle at M. Clearly, M is the midpoint of the arc BMC. The line BM clearly contains the incenter I and the excenter I_a.

Lemma 4. $MB = MI = MI_a = MC$.

[1] The referee has pointed out that these results had been known earlier, and can be found, for example, in the nineteenth century work of John Casey [1].

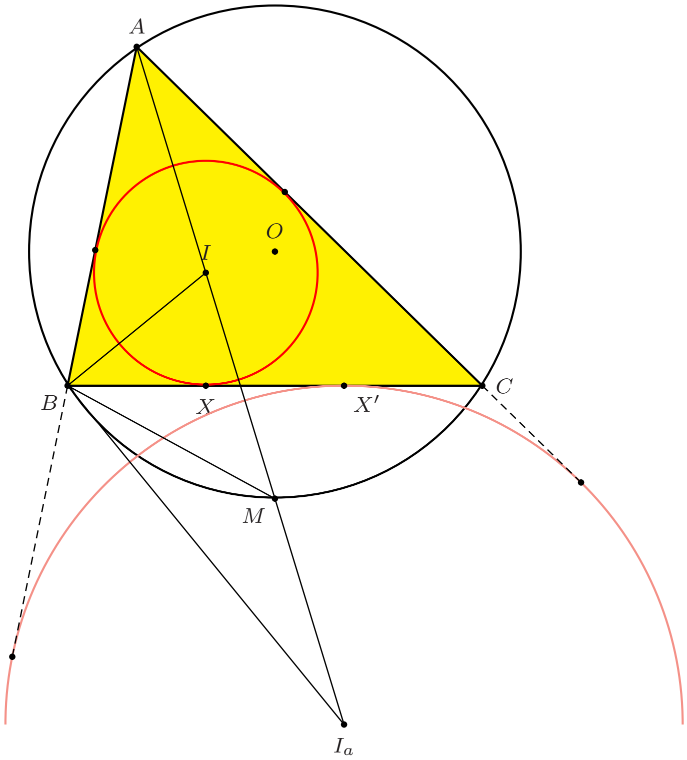

Figure 3. $r_a + r_b + r_c = 4R + r$

Proof. It is enough to prove that $MB = MI$. See Figure 3. This follows by an easy calculation of angles.
(i) $\angle IBI_a = 90°$ since the two bisectors of angle B are perpendicular to each other.
(ii) The midpoint N of I_aI is the circumcenter of triangle IBI_a, so $NB = NI = NI_a$.
(iii) From the circle (IBI_a) we see $\angle BNA = \angle BNI = 2\angle BCI = \angle BCA$, but this means that N lies on the circumcircle (ABC) and thus coincides with M.
It follows that $MI_a = MB = MI$, and M is the midpoint of II_a.

The same reasoning shows that $MC = MI = MI_a$ as well. □

Now, let I' be the intersection of the line IO and the perpendicular from I_a to BC. See Figure 4. Note that this latter line is parallel to OM. Since M is the midpoint of II_a, O is the midpoint of II'. It follows that I' is the reflection of I in O. Also, $I'I_a = 2 \cdot OM = 2R$. Similarly, $I'I_b = I'I_c = 2R$. We summarize this in the following proposition.

Proposition 5. *The circle through the three excenters has radius $2R$ and center I', the reflection of I in O.*

Remark. Proposition 5 also follows from the fact that the circumcircle is the nine point circle of triangle $I_aI_bI_c$, and I is the orthocenter of this triangle.

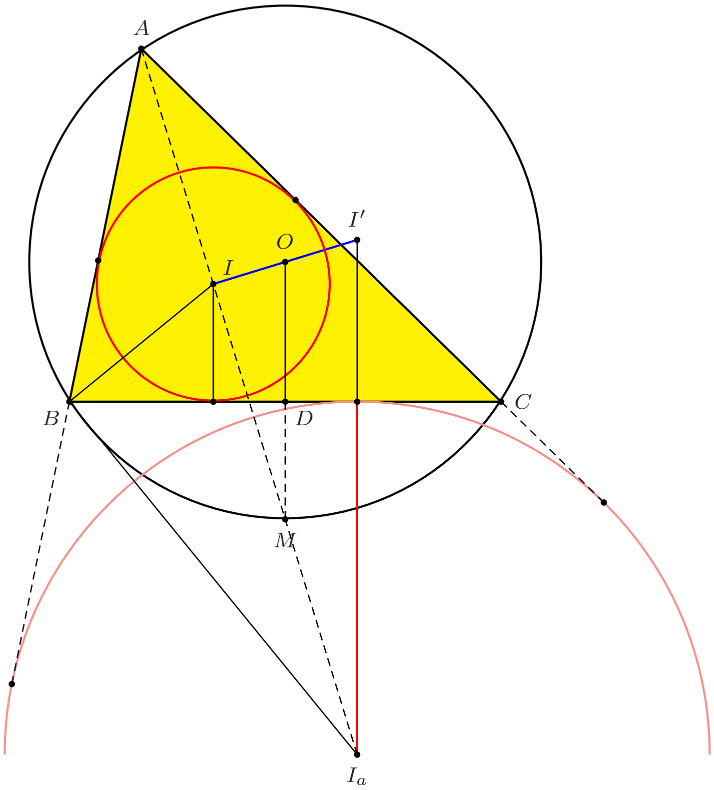

Figure 4. $I'I_a = 2R$

Proposition 6. $r_a + r_b + r_c = 4R + r$.

Proof. The line I_aI' intersects BC at the point X' of tangency with the excircle. Note that $I'X' = 2R - r_a$. Since O is the midpoint of II', we have $IX + I'X' = 2 \cdot OD$. From this, we have

$$2 \cdot OD = r + (2R - r_a). \tag{2}$$

Consider the excenters I_b and I_c. Since the angles I_bBI_c and I_bCI_c are both right angles, the four points I_b, I_c, B, C are on a circle, whose center is the midpoint N of I_bI_c. See Figure 5. The center N must lie on the perpendicular bisector of BC, which is the line OM. Therefore N is the antipodal point of M on the circumcircle, and we have $2ND = r_b + r_c$. Thus, $2(R + OD) = r_b + r_c$. From (2), we have $r_a + r_b + r_c = 4R + r$. □

4. Proof of Theorem 2

We are now ready to prove Theorem 2.

(1) Since $AH = 2 \cdot OD$, by (2) we express this in terms of R, r and r_a; similarly for BH and CH:

$$AH = 2R + r - r_a, \qquad BH = 2R + r - r_b, \qquad CH = 2R + r - r_c.$$

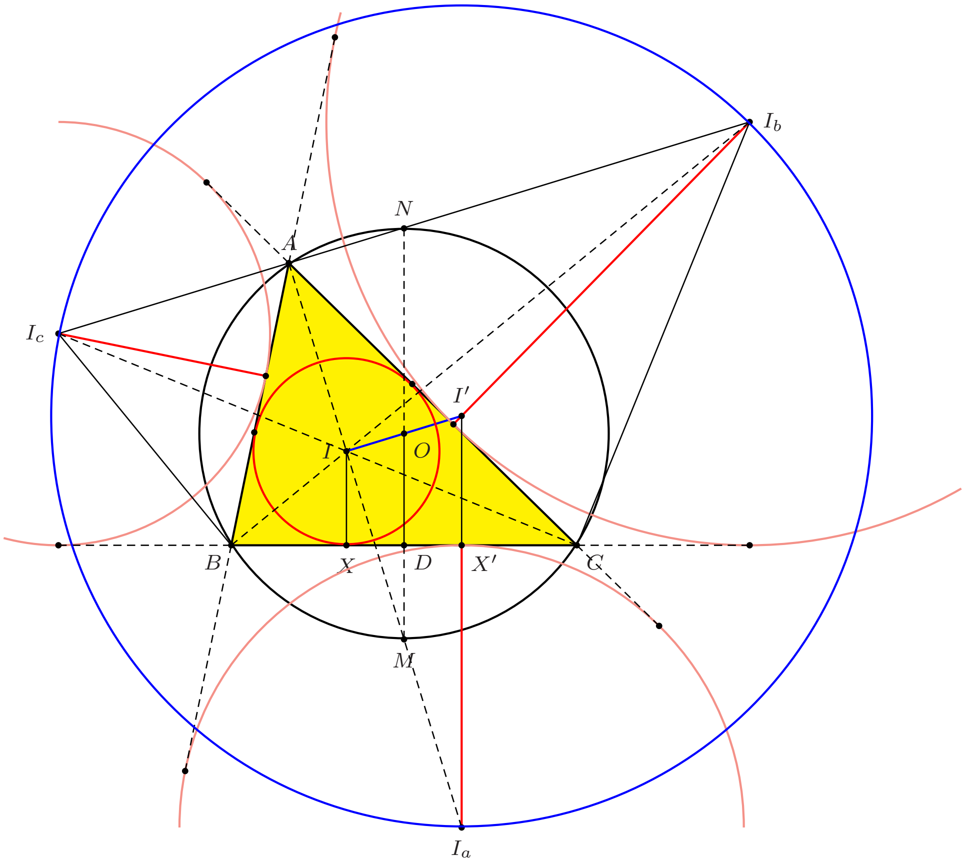

Figure 5. $r_a + r_b + r_c = 4R + r$

From these,

$$\begin{aligned}AH + BH + CH + 2R =& 8R + 3r - (r_a + r_b + r_c)\\ =& 2(4R + r) + r - (r_a + r_b + r_c)\\ =& 2(r_a + r_b + r_c) + r - (r_a + r_b + r_c)\\ =& r_a + r_b + r_c + r.\end{aligned}$$

(2) This follows from simple calculation making use of Proposition 6.

$$\begin{aligned}& AH^2 + BH^2 + CH^2 + (2R)^2\\ =& (2R + r - r_a)^2 + (2R + r - r_b)^2 + (2R + r - r_c)^2 + 4R^2\\ =& 3(2R + r)^2 - 2(2R + r)(r_a + r_b + r_c) + r_a^2 + r_b^2 + r_c^2 + 4R^2\\ =& 3(2R + r)^2 - 2(2R + r)(4R + r) + 4R^2 + r_a^2 + r_b^2 + r_c^2\\ =& r^2 + r_a^2 + r_b^2 + r_c^2.\end{aligned}$$

This completes the proof of Theorem 2.

5. Converse of Hansen's theorem

We prove a strong converse of Hansen's theorem (Theorem 10 below).

Proposition 7. *A triangle ABC satisfies*

$$r_a^2 + r_b^2 + r_c^2 + r^2 = a^2 + b^2 + c^2 \tag{3}$$

if and only if it contains a right angle.

Proof. Using $AH = 2R\cos A$ and $a = 2R\sin A$, and similar expressions for BH, CH, b, and c, we have

$$\begin{aligned}
&AH^2 + BH^2 + CH^2 + (2R)^2 - (a^2 + b^2 + c^2)\\
=&4R^2(\cos^2 A + \cos^2 B + \cos^2 C + 1 - \sin^2 A - \sin^2 B - \sin^2 C)\\
=&4R^2(2\cos^2 A + \cos 2B + \cos 2C)\\
=&8R^2(\cos^2 A + \cos(B+C)\cos(B-C))\\
=&-8R^2\cos A(\cos(B+C) + \cos(B-C))\\
=&-16R^2\cos A\cos B\cos C.
\end{aligned}$$

By Theorem 2(2), the condition (3) holds if and only if $AH^2 + BH^2 + CH^2 + (2R)^2 = a^2 + b^2 + c^2$. One of $\cos A$, $\cos B$, $\cos C$ must be zero from above. This means that triangle ABC contains a right angle. □

In the following lemma we collect some useful and well known results. They can be found more or less directly in [3].

Lemma 8. (1) $r_ar_b + r_br_c + r_cr_a = s^2$.
(2) $r_a^2 + r_b^2 + r_c^2 = (4R + r)^2 - 2s^2$.
(3) $ab + bc + ca = s^2 + (4R + r)r$.
(4) $a^2 + b^2 + c^2 = 2s^2 - 2(4R + r)r$.

Proof. (1) follows from the formulas for the exradii and the Heron formula.

$$\begin{aligned}
r_ar_b + r_br_c + r_cr_a =&\frac{\triangle^2}{(s-a)(s-b)} + \frac{\triangle^2}{(s-b)(s-c)} + \frac{\triangle^2}{(s-c)(s-a)}\\
=&s((s-c) + (s-a) + (s-b))\\
=&s^2.
\end{aligned}$$

From this (2) easily follows.

$$\begin{aligned}
r_a^2 + r_b^2 + r_c^2 =&(r_a + r_b + r_c)^2 - 2(r_ar_b + r_br_c + r_cr_a)\\
=&(4R + r)^2 - 2s^2.
\end{aligned}$$

Again, by Proposition 6,

$$\begin{aligned}
&4R+r\\
=&r_a+r_b+r_c\\
=&\frac{\triangle}{s-a}+\frac{\triangle}{s-b}+\frac{\triangle}{s-c}\\
=&\frac{\triangle}{(s-a)(s-b)(s-c)}\left((s-b)(s-c)+(s-c)(s-a)+(s-a)(s-b)\right)\\
=&\frac{1}{r}\left(3s^2-2(a+b+c)s+(ab+bc+ca)\right)\\
=&\frac{1}{r}\left((ab+bc+ca)-s^2\right).
\end{aligned}$$

An easy rearrangement gives (3).

(4) follows from (3) since $a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)=4s^2-2(s^2+(4R+r)r)=2s^2-2(4R+r)r$. □

Proposition 9. $r_a^2+r_b^2+r_c^2+r^2=a^2+b^2+c^2$ *if and only if* $2R+r=s$.

Proof. By Lemma 8(2) and (4), $r_a^2+r_b^2+r_c^2+r^2=a^2+b^2+c^2$ if and only if $(4R+r)^2-2s^2+r^2=2s^2-2(4R+r)r$; $4s^2=(4R+r)^2+2(4R+r)r+r^2=(4R+2r)^2=4(2R+r)^2$; $s=2R+r$. □

Theorem 10. *The following statements for a triangle ABC are equivalent.*
(1) $r_a+r_b+r_c+r=a+b+c$.
(2) $r_a^2+r_b^2+r_c^2+r^2=a^2+b^2+c^2$.
(3) $R+2r=s$.
(4) *One of the angles is a right angle.*

Proof. (1) $\Longrightarrow$ (3): This follows easily from Proposition 6.

(3) $\Longleftrightarrow$ (2): Proposition 9 above.

(2) $\Longleftrightarrow$ (4): Proposition 7 above.

(4) $\Longrightarrow$ (1): Theorem 1 (1). □

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