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Some Constructions Related to the Kiepert Hyperbola

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Abstract. Given a reference triangle and its Kiepert hyperbola $\mathcal{K}$, we study several construction problems related to the triangles which have $\mathcal{K}$ as their own Kiepert hyperbolas. Such triangles necessarily have their vertices on $\mathcal{K}$, and are called special Kiepert inscribed triangles. Among other results, we show that the family of special Kiepert inscribed triangles all with the same centroid G form part of a poristic family between $\mathcal{K}$ and an inscribed conic with center which is the inferior of the Kiepert center.

1. Special Kiepert inscribed triangles

Given a triangle ABC and its Kiepert hyperbola $\mathcal{K}$, consisting of the Kiepert perspectors

$$K(t) = \left(\frac{1}{S_A+t} : \frac{1}{S_B+t} : \frac{1}{S_C+t}\right), \qquad t \in \mathbb{R} \cup \{\infty\},$$

we study triangles with vertices on $\mathcal{K}$ having $\mathcal{K}$ as their own Kiepert hyperbolas. We shall work with homogeneous barycentric coordinates and make use of standard notations of triangle geometry as in [2]. Basic results on triangle geometry can be found in [3]. The Kiepert hyperbola has equation

$$K(x,y,z) := (S_B - S_C)yz + (S_C - S_A)zx + (S_A - S_B)xy = 0 \qquad (1)$$

in homogeneous barycentric coordinates. Its center, the Kiepert center

$$K_{\rm i} = ((S_B - S_C)^2 : (S_C - S_A)^2 : (S_A - S_B)^2),$$

lies on the Steiner inellipse. In this paper we shall mean by a Kiepert inscribed triangle one whose vertices are on the Kiepert hyperbola $\mathcal{K}$. If a Kiepert inscribed triangle is perspective with ABC, it is called the Kiepert cevian triangle of its perspector. Since the Kiepert hyperbola of a triangle can be characterized as the rectangular circum-hyperbola containing the centroid, our objects of interest are Kiepert inscribed triangles whose centroids are Kiepert perspectors. We shall assume the vertices to be finite points on $\mathcal{K}$, and call such triangles special Kiepert inscribed triangles. We shall make frequent use of the following notations.

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$$\begin{aligned} P(t) &= ((S_B - S_C)(S_A + t) : (S_C - S_A)(S_B + t) : (S_A - S_B)(S_C + t)) \\ Q(t) &= ((S_B - S_C)^2(S_A + t) : (S_C - S_A)^2(S_B + t) : (S_A - S_B)^2(S_C + t)) \\ f_2 &= S_{AA} + S_{BB} + S_{CC} - S_{BC} - S_{CA} - S_{AB} \\ f_3 &= S_A(S_B - S_C)^2 + S_B(S_C - S_A)^2 + S_C(S_A - S_B)^2 \\ f_4 &= (S_{AA} - S_{BC})S_{BC} + (S_{BB} - S_{CA})S_{CA} + (S_{CC} - S_{AB})S_{AB} \\ g_3 &= (S_A - S_B)(S_B - S_C)(S_C - S_A) \end{aligned}$$

Here, $P(t)$ is a typical infinite point, and $Q(t)$ is a typical point on the tangent of the Steiner inellipse through $K_{\rm i}$. For $k = 2, 3, 4$, the function f_k, is a symmetric function in S_A, S_B, S_C of degree k.

Proposition 1. *The area of a triangle with vertices $K(t_i)$, $i = 1, 2, 3$, is*

$$\left|\frac{g_3(t_1-t_2)(t_2-t_3)(t_3-t_1)}{\prod(S^2+2(S_A+S_B+S_C)t_i+3t_i^2}\right| \cdot \triangle ABC.$$

Proposition 2. *A Kiepert inscribed triangle with vertices $K(t_i)$, $i = 1, 2, 3$, is special,* i.e.*, with centroid on the Kiepert hyperbola, if and only if*

$$S^2 f_2' + (S_A + S_B + S_C) f_3' - 3 f_4' = 0,$$

where f_2', f_3', f_4' are the functions f_2, f_3, f_4 with S_A, S_B, S_C replaced by t_1, t_2, t_3.

We shall make use of the following simple construction.

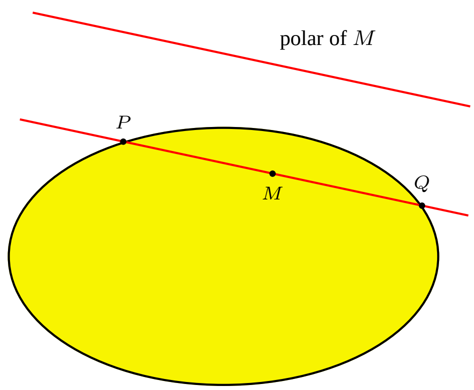

Figure 1. Construction of chord of conic with given midpoint

Construction 3. *Given a conic $\mathcal{C}$ and a point M, to construct the chord of $\mathcal{C}$ with M as midpoint, draw*
(i) *the polar of M with respect to $\mathcal{C}$,*
(ii) *the parallel through M to the line in* (i)*.*
If the line in (ii) *intersects $\mathcal{C}$ at the two real points P and Q, then the midpoint of PQ is M.*

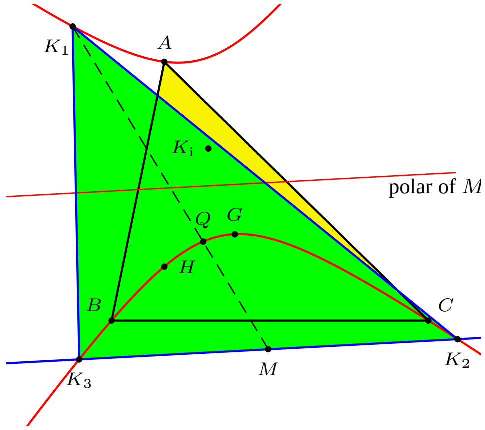

Figure 2. Construction of Kiepert inscribed triangle with prescribed centroid and one vertex

A simple application of Construction 3 gives a Kiepert inscribed triangle with prescribed centroid Q and one vertex K_1: simply take M to be the point dividing K_1Q in the ratio $K_1M : MQ = 3 : -1$. See Figure 2.

Here is an interesting family of Kiepert inscribed triangles with prescribed centroids on $\mathcal{K}$.

Construction 4. *Given a Kiepert perspector $K(t)$, construct*
(i) *K_1 on $\mathcal{K}$ and M such that the segment K_1M is trisected at $K_{\rm i}$ and $K(t)$,*
(ii) *the parallel through M to the tangent of $\mathcal{K}$ at $K(t)$,*
(iii) *the intersections K_2 and K_3 of $\mathcal{K}$ with the line in* (ii).
Then $K_1K_2K_3$ is a special Kiepert inscribed triangle with centroid $K(t)$. See Figure 3.

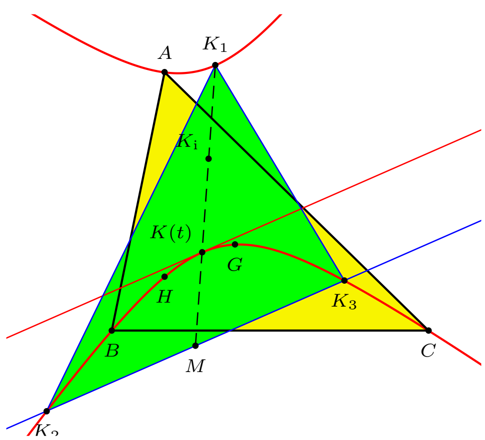

Figure 3. Kiepert inscribed triangle with centroid $K(t)$

It is interesting to note that the area of the Kiepert inscribed triangle is independent of t. It is $\frac{3\sqrt{3}}{2}|g_3|f_2^{-\frac{3}{2}}$ times that of triangle ABC. This result and many others in the present paper are obtained with the help of a computer algebra system.

2. Special Kiepert cevian triangles

Given a point $P = (u : v : w)$, the vertices of its Kiepert cevian triangle are

$$A_P = \left(\frac{-(S_B - S_C)vw}{(S_A - S_B)v + (S_C - S_A)w} : v : w\right),$$
$$B_P = \left(u : \frac{-(S_C - S_A)wu}{(S_B - S_C)w + (S_A - S_B)u} : w\right),$$
$$C_P = \left(u : v : \frac{-(S_A - S_B)uv}{(S_C - S_A)u + (S_B - S_C)v}\right).$$

These are Kiepert perspectors with parameters t_A, t_B, t_C given by

$$t_A = -\frac{S_Bv - S_Cw}{v - w}, \qquad t_B = -\frac{S_Cw - S_Au}{w - u}, \qquad t_C = -\frac{S_Au - S_Bv}{u - v}.$$

Clearly, if P is on the Kiepert hyperbola, the Kiepert cevian triangle $A_PB_PC_P$ degenerates into the point P.

Theorem 5. *The centroid of the Kiepert cevian triangle of P lies on the Kiepert hyperbola if and only if P is*
(i) *an infinite point, or*
(ii) *on the tangent at $K_{\rm i}$ to the Steiner inellipse.*

Proof. Let $P = (u : v : w)$ in homogeneous barycentric coordinates. Applying Proposition 2, we find that the centroid of $A_PB_PC_P$ lies on the Kiepert hyperbola if and only if

$$(u + v + w)K(u, v, w)^2L(u, v, w)P(u, v, w) = 0,$$

where

$$L(u, v, w) = \frac{u}{S_B - S_C} + \frac{v}{S_C - S_A} + \frac{w}{S_A - S_B},$$
$$P(u, v, w) = \prod((S_A - S_B)v^2 - 2(S_B - S_C)vw + (S_C - S_A)w^2).$$

The factors $u + v + w$ and $K(u, v, w)$ clearly define the line at infinity and the Kiepert hyperbola $\mathcal{K}$ respectively. On the other hand, the factor $L(u, v, w)$ defines the line

$$\frac{x}{S_B - S_C} + \frac{y}{S_C - S_A} + \frac{z}{S_A - S_B} = 0, \tag{2}$$

which is the tangent of the Steiner inellipse at $K_{\rm i}$.

Each factor of $P(u, v, w)$ defines two points on a sideline of triangle ABC. If we set $(x, y, z) = (-(v + w), v, w)$ in (1), the equation reduces to $(S_A - S_B)v^2 - 2(S_B - S_C)vw + (S_C - S_A)w^2$. This shows that the two points on the line BC are the intercepts of lines through A parallel to the asymptotes of $\mathcal{K}$, and the corresponding Kiepert cevian triangles have vertices at infinite points. This is similarly the case for the other two factors of $P(u, v, w)$. □

Remark. Altogether, the six points defined by $P(u,v,w)$ above determine a conic with equation

$$G(x,y,z)=\sum\frac{x^2}{S_B-S_C}-\frac{2(S_B-S_C)yz}{(S_C-S_A)(S_A-S_B)}=0.$$

Since

$$\begin{aligned}&g_3\cdot G(x,y,z)\\=&-f_2(x+y+z)^2+\sum(S_B-S_C)^2x^2-2(S_C-S_A)(S_A-S_B)yz,\end{aligned}$$

this conic is a translation of the inscribed conic

$$\sum(S_B-S_C)^2x^2-2(S_C-S_A)(S_A-S_B)yz=0,$$

which is the Kiepert parabola. See Figure 4.

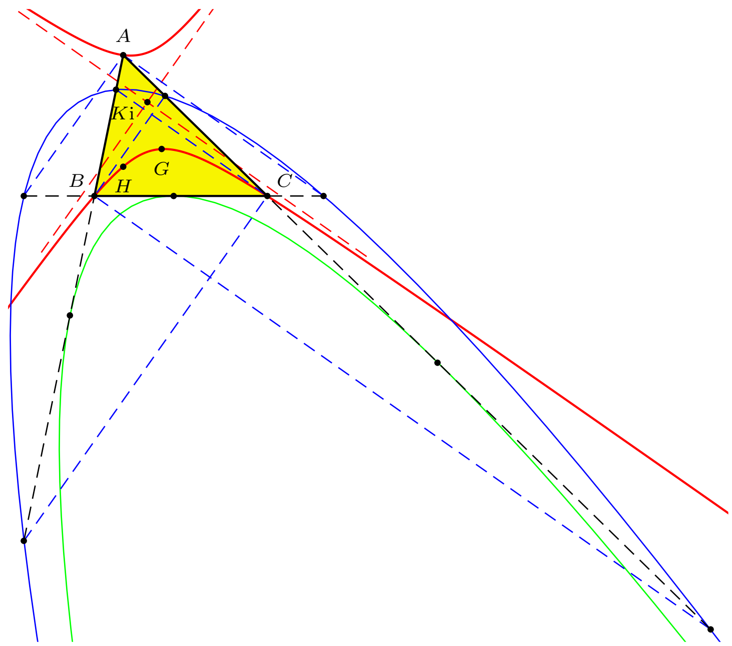

Figure 4. Translation of Kiepert parabola

3. Kiepert cevian triangles of infinite points

Consider a typical infinite point

$$P(t)=((S_B-S_C)(S_A+t):(S_C-S_A)(S_B+t):(S_A-S_B)(S_C+t))$$

in homogeneous barycentric coordinates. It can be easily verified that $P(t)$ is the infinite point of perpendiculars to the line joining the Kiepert perspector $K(t)$ to the orthocenter H.[1] The Kiepert cevian triangle of $P(t)$ has vertices

[1] This is the line $\sum S_A(S_B-S_C)(S_A+t)x=0$.

$$A(t) = \left(\frac{(S_B - S_C)(S_B + t)(S_C + t)}{S_B + S_C + 2t} : (S_C - S_A)(S_B + t) : (S_A - S_B)(S_C + t)\right),$$
$$B(t) = \left((S_B - S_C)(S_A + t) : \frac{(S_C - S_A)(S_C + t)(S_A + t)}{S_C + S_A + 2t} : (S_A - S_B)(S_C + t)\right),$$
$$C(t) = \left((S_B - S_C)(S_A + t) : (S_C - S_A)(S_B + t) : \frac{(S_A - S_B)(S_A + t)(S_B + t)}{S_C + S_A + 2t}\right).$$

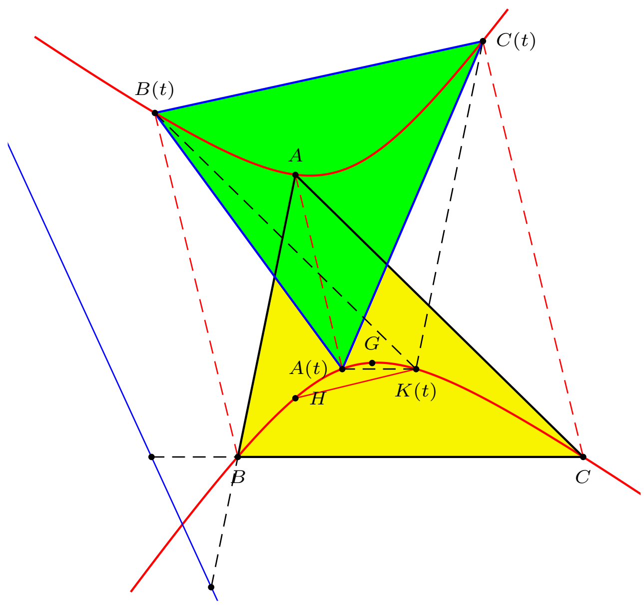

Figure 5. The Kiepert cevian triangle of $P(t)$ is the same as the Kiepert parallelian triangle of $K(t)$

It is also true that the line joining $A(t)$ to $K(t)$ is parallel to BC;[2] similarly for $B(t)$ and $C(t)$. Thus, we say that the Kiepert cevian triangle of the infinite point $P(t)$ is the same as the Kiepert parallelian triangle of the Kiepert perspector $K(t)$. See Figure 5. It is interesting to note that the area of triangle $A(t)B(t)C(t)$ is equal to that of triangle ABC, but the triangles have opposite orientations.

Now, the centroid of triangle $A(t)B(t)C(t)$ is the point

$$\left(\frac{S_B - S_C}{S_{AB} + S_{AC} - 2S_{BC} - (S_B + S_C - 2S_A)t} : \cdots : \cdots\right),$$

which, by Theorem 5, is a Kiepert perspector. It is $K(s)$ where s is given by

$$2f_2 \cdot st + f_3 \cdot (s + t) - 2f_4 = 0. \tag{3}$$

Proposition 6. *Two distinct Kiepert perspectors have parameters satisfying* (3) *if and only if the line joining them is parallel to the orthic axis.*

[2]This is the line $-(S_A + t)(S_B + S_C + 2t)x + (S_B + t)(S_C + t)(y + z) = 0$.

Proof. The orthic axis $S_Ax + S_By + S_Cz = 0$ has infinite point

$$P(\infty) = (S_B - S_C : S_C - S_A : S_A - S_B).$$

The line joining $K(s)$ and $K(t)$ is parallel to the orthic axis if and only if

$$\begin{vmatrix} \frac{1}{S_A+s} & \frac{1}{S_B+s} & \frac{1}{S_C+s} \\ \frac{1}{S_A+t} & \frac{1}{S_B+t} & \frac{1}{S_C+t} \\ S_B - S_C & S_C - S_A & S_A - S_B \end{vmatrix} = 0.$$

For $s \neq t$, this is the same condition as (3). □

This leads to the following construction.

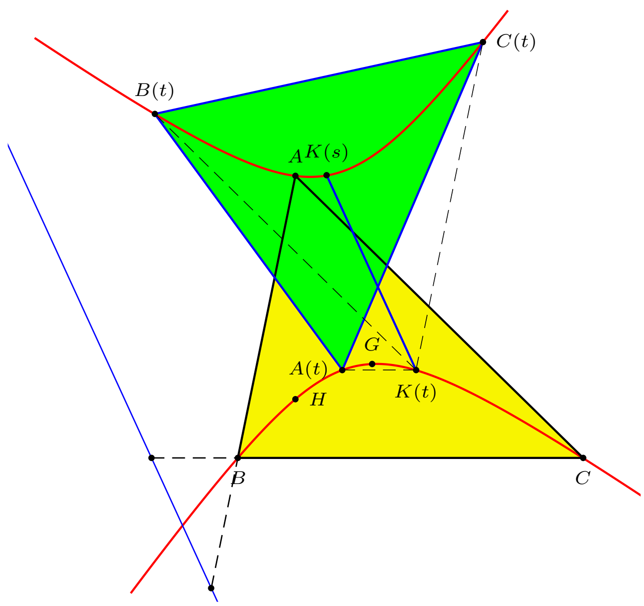

Figure 6. The Kiepert cevian triangle of $P(t)$ has centroid $K(s)$

Construction 7. *Given a Kiepert perspector $K(s)$, to construct a Kiepert cevian triangle with centroid $K(s)$, draw*
(i) *the parallel through $K(s)$ to the orthic axis to intersect the Kiepert hyperbola again at $K(t)$,*
(ii) *the parallels through $K(t)$ to the sidelines of the triangle to intersect $\mathcal{K}$ again at $A(t)$, $B(t)$, $C(t)$ respectively.*
Then, $A(t)B(t)C(t)$ has centroid $K(s)$. See Figure 6.

4. Special Kiepert inscribed triangles with common centroid G

We construct a family of Kiepert inscribed triangles with centroid G, the centroid of the reference triangle ABC. This can be easily accomplished with the help of Construction 3. Beginning with a Kiepert perspector $K_1 = K(t)$ and $Q = G$, we easily determine

$$M = ((S_A+t)(S_B+S_C+2t) : (S_B+t)(S_C+S_A+2t) : (S_C+t)(S_A+S_B+2t)).$$

The line through M parallel to its own polar with respect to $\mathcal{K}$[3] has equation

$$\frac{S_B - S_C}{S_A + t}x + \frac{S_C - S_A}{S_B + t}y + \frac{S_A - S_B}{S_C + t}z = 0. \tag{4}$$

As t varies, this line envelopes the conic

$$\begin{aligned}&(S_B - S_C)^4x^2 + (S_C - S_A)^4y^2 + (S_A - S_B)^4z^2\\ &- 2(S_B - S_C)^2(S_C - S_A)^2xy - 2(S_C - S_A)^2(S_A - S_B)^2yz\\ &- 2(S_A - S_B)^2(S_B - S_C)^2zx = 0,\end{aligned}$$

which is the inscribed ellipse $\mathcal{E}$ tangent to the sidelines of ABC at the traces of

$$\left(\frac{1}{(S_B - S_C)^2} : \frac{1}{(S_C - S_A)^2} : \frac{1}{(S_A - S_B)^2}\right),$$

and to the Kiepert hyperbola at G, and to the line (4) at the point

$$((S_A + t)^2 : (S_B + t)^2 : (S_C + t)^2).$$

It has center

$$((S_C-S_A)^2+(S_A-S_B)^2 : (S_A-S_B)^2+(S_B-S_C)^2 : (S_B-S_C)^2+(S_C-S_A)^2),$$

the inferior of the Kiepert center K_i. See Figure 7.

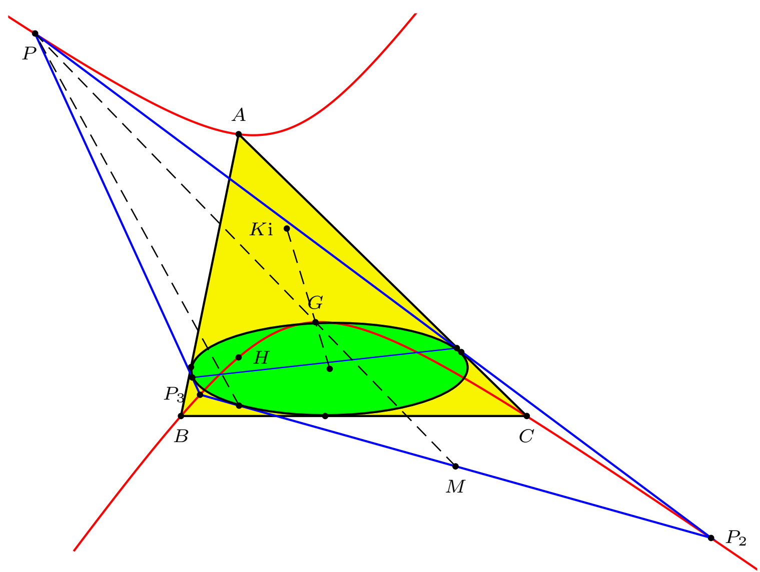

Figure 7. Poristic triangles with common centroid G

[3]The polar of M has equation $\sum(S_B - S_C)(S_{AA} - S^2 - 2(S_B + S_C)t - 2t^2)x = 0$ and has infinite point $((S_A + t)(S_A(S_B + S_C - 2t) - (S_B + S_C)(S_B - S_C + t)) : \cdots : \cdots)$.

Theorem 8. *A poristic triangle completed from a point on the Kiepert hyperbola outside the inscribed ellipse $\mathcal{E}$ (with center the inferior of K_i) has its center at G and therefore has $\mathcal{K}$ as its Kiepert hyperbola.*

More generally, if we replace G by a Kiepert perspector K_g, the envelope is a conic with center which divides K_iK_g in the ratio $3 : -1$. It is an ellipse inscribed in the triangle in Construction 4.

5. A family of special Kiepert cevian triangles

5.1. *Triple perspectivity.* According to Theorem 5, there is a family of special Kiepert cevian triangles with perspectors on the line (2) which is the tangent of the Steiner inellipse at K_i. Since this line also contains the Jerabek center

$$J_e = (S_A(S_B - S_C)^2 : S_B(S_C - S_A)^2 : S_C(S_A - S_B)^2),$$

its points can be parametrized as

$$Q(t) = ((S_B - S_C)^2(S_A + t) : (S_C - S_A)^2(S_B + t) : (S_A - S_B)^2(S_C + t)).$$

The Kiepert cevian triangle of $Q(t)$ has vertices

$$A'(t) = \left(\frac{(S_C - S_A)(S_A - S_B)(S_B + t)(S_C + t)}{S_A + t} : (S_C - S_A)^2(S_B + t) : (S_A - S_B)^2(S_C + t)\right),$$
$$B'(t) = \left((S_B - S_C)^2(S_A + t) : \frac{(S_A - S_B)(S_B - S_C)(S_C + t)(S_A + t)}{S_B + t} : (S_A - S_B)^2(S_C + t)\right),$$
$$C'(t) = \left((S_B - S_C)^2(S_A + t) : (S_C - S_A)^2(S_B + t) : \frac{(S_B - S_C)(S_C - S_A)(S_A + t)(S_B + t)}{S_C + t}\right).$$

Theorem 9. *The Kiepert cevian triangle of $Q(t)$ is triply perspective to ABC. The three perspectors are collinear on the tangent of the Steiner inellipse at K_i.*

Proof. The triangles $B'(t)C'(t)A'(t)$ and $C'(t)A'(t)B'(t)$ are each perspective to ABC, at the points

$$Q'(t) = \left(\frac{S_C + t}{S_C - S_A} : \frac{S_A + t}{S_A - S_B} : \frac{S_B + t}{S_B - S_C}\right),$$

and

$$Q''(t) = \left(\frac{S_B + t}{S_A - S_B} : \frac{S_C + t}{S_B - S_C} : \frac{S_A + t}{S_C - S_A}\right)$$

respectively. These two points are clearly on the line (2). □

5.2. *Special Kiepert cevian triangles with the same area as ABC.* The area of triangle $A'(t)B'(t)C'(t)$ is

$$\frac{(f_2 \cdot t^2 + f_3 \cdot t - f_4)^3}{\prod (f_2 \cdot (S_A + t)^2 - (S_C - S_A)^2(S_A - S_B)^2)}$$

Among these, four have the same area as the reference triangle.

5.2.1. $t = \frac{S_A(S_B+S_C)-2S_{BC}}{S_B+S_C-2S_A}$. The points

$$\begin{aligned}Q(t) =&(-2(S_B - S_C) : S_C - S_A : S_A - S_B),\\ Q'(t) =&(S_B - S_C : -2(S_C - S_A) : S_A - S_B),\\ Q''(t) =&(S_B - S_C : S_C - S_A : -2(S_A - S_B)),\end{aligned}$$

give the Kiepert cevian triangle

$$\begin{aligned}A'_1 =&(-(S_B - S_C) : 2(S_C - S_A) : 2(S_A - S_B)),\\ B'_1 =&(2(S_B - S_C) : -(S_C - S_A) : 2(S_A - S_B)),\\ C'_1 =&(2(S_B - S_C) : 2(S_C - S_A) : -(S_A - S_B)).\end{aligned}$$

This has centroid

$$K\left(-\frac{f_3}{2f_2}\right) = \left(\frac{S_B - S_C}{S_B + S_C - 2S_A} : \frac{S_C - S_A}{S_C + S_A - 2S_B} : \frac{S_A - S_B}{S_A + S_B - 2S_C}\right).$$

$A'(t)B'(t)C'(t)$ is also the Kiepert cevian triangle of the infinite point $P(\infty)$ (of the orthic axis). See Figure 8.

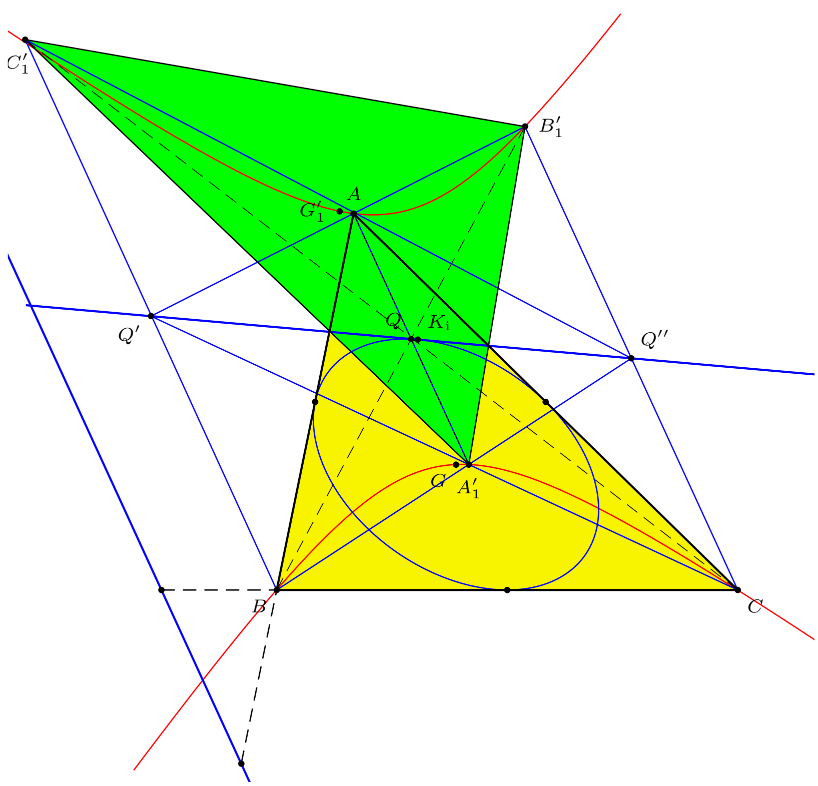

Figure 8. Oppositely oriented triangle triply perspective with ABC at three points on tangent at K_i

5.2.2. $t = \infty$. With the Kiepert center $K_{\rm i} = Q(\infty)$, we have the points

$$Q(\infty) =((S_B - S_C)^2 : (S_C - S_A)^2 : (S_A - S_B)^2),$$
$$Q'(\infty) = \left(\frac{1}{S_A - S_B} : \frac{1}{S_B - S_C} : \frac{1}{S_C - S_A}\right),$$
$$Q''(\infty) = \left(\frac{1}{S_C - S_A} : \frac{1}{S_A - S_B} : \frac{1}{S_B - S_C}\right),$$

The points $Q'(\infty)$ and $Q''(\infty)$ are the intersection with the parallels through B, C to the line joining A to the Steiner point $S_{\rm t} = \left(\frac{1}{S_B-S_C} : \frac{1}{S_C-S_A} : \frac{1}{S_A-S_B}\right)$. These points give the Kiepert cevian triangle which is the image of ABC under the homothety $\mathsf{h}(K_{\rm i}, -1)$:

$$A_2' =((S_C - S_A)(S_A - S_B) : (S_C - S_A)^2 : (S_A - S_B)^2),$$
$$B_2' =((S_B - S_C)^2 : (S_A - S_B)(S_B - S_C) : (S_A - S_B)^2),$$
$$C_2' =((S_B - S_C)^2 : (S_C - S_A)^2 : (S_C - S_A)(S_B - S_C)),$$

which has centroid

$$K\left(-\frac{S_A + S_B + S_C}{3}\right) = \left(\frac{1}{S_B + S_C - 2S_A} : \frac{1}{S_C + S_A - 2S_B} : \frac{1}{S_A + S_B - 2S_C}\right).$$

The points $Q'(t)$, $Q''(t)$ and G_2' are on the Steiner circum-ellipse. See Figure 9.

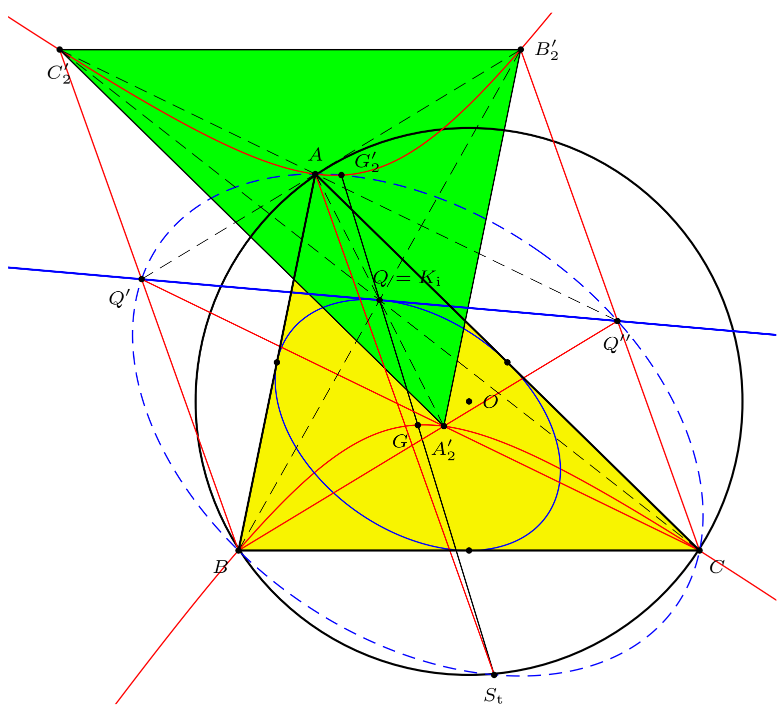

Figure 9. Oppositely congruent triangle triply perspective with ABC at three points on tangent at $K_{\rm i}$

5.2.3. $t = \frac{-f_3}{2f_2}$. $Q(t)$ is the infinite point of the line (2).

$$Q(t) = ((S_B - S_C)(S_B + S_C - 2S_A) : (S_C - S_A)(S_C + S_A - 2S_B) : (S_A - S_B)(S_A + S_B - 2S_C)),$$
$$Q'(t) = ((S_B - S_C)(S_A + S_B - 2S_C) : (S_C - S_A)(S_B + S_C - 2S_A) : (S_A - S_B)(S_C + S_A - 2S_B)),$$
$$Q''(t) = ((S_B - S_C)(S_C + S_A - 2S_B) : (S_C - S_A)(S_A + S_B - 2S_C) : (S_A - S_B)(S_B + S_C - 2S_A)).$$

These give the Kiepert cevian triangle

$$A_3' = \left(\frac{S_B - S_C}{S_B + S_C - 2S_A} : \frac{S_C - S_A}{S_A + S_B - 2S_C} : \frac{S_A - S_B}{S_C + S_A - 2S_B}\right),$$
$$B_3' = \left(\frac{S_B - S_C}{S_A + S_B - 2S_C} : \frac{S_C - S_A}{S_C + S_A - 2S_B} : \frac{S_A - S_B}{S_B + S_C - 2S_A}\right),$$
$$C_3' = \left(\frac{S_B - S_C}{S_C + S_A - 2S_B} : \frac{S_C - S_A}{S_B + S_C - 2S_A} : \frac{S_A - S_B}{S_A + S_B - 2S_C}\right),$$

with centroid

$$\left(\frac{S_B - S_C}{(S_B - S_C)^2 + 2(S_C - S_A)(S_A - S_B)} : \cdots : \cdots\right).$$

See Figure 10.

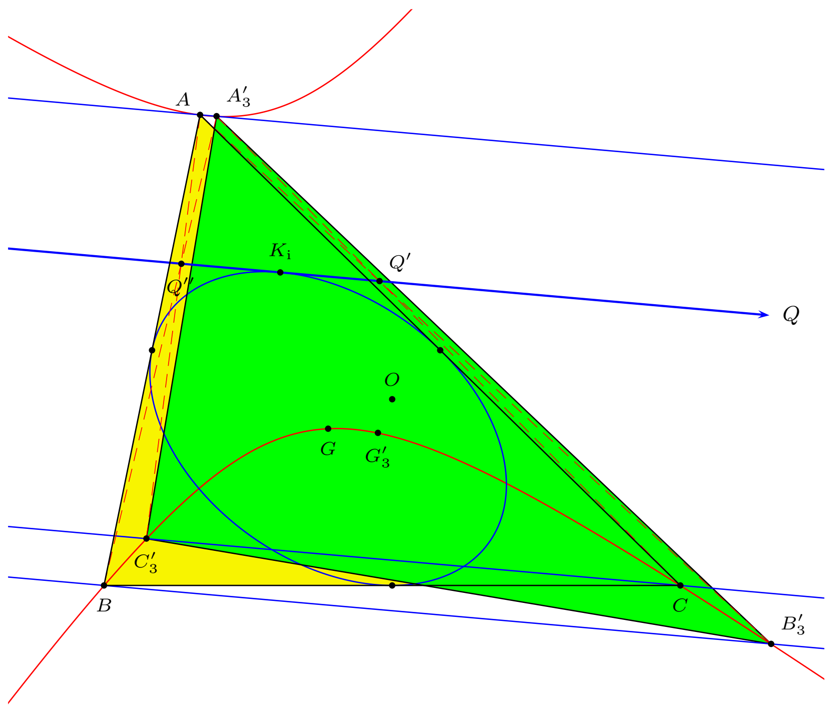

Figure 10. Triangle triply perspective with ABC (with the same orientation) at three points on tangent at K_i

5.2.4. $t = -S_A$. For $t = -S_A$, we have

$$\begin{aligned} Q(t) =&(0 : S_C - S_A : -(S_A - S_B)), \\ Q'(t) =&(-(S_B - S_C) : 0 : S_A - S_B), \\ Q''(t) =&(S_B - S_C : -(S_C - S_A) : 0). \end{aligned}$$

These points are the intercepts Q_a, Q_b, Q_c of the line (2) with the sidelines BC, CA, AB respectively. The lines AQ_a, BQ_b, CQ_c are the tangents to $\mathcal{K}$ at the vertices. The common Kiepert cevian triangle of Q_a, Q_b, Q_c is ABC oppositely oriented as ACB, CBA, BAC, triply perspective with ABC at Q_a, Q_b, Q_c respectively.

6. Special Kiepert inscribed triangles with two given vertices

Construction 10. *Given two points K_1 and K_2 on the Kiepert hyperbola $\mathcal{K}$, construct*
(i) *the midpoint M of K_1K_2,*
(ii) *the polar of M with respect to $\mathcal{K}$,*
(iii) *the reflection of the line K_1K_2 in the polar in* (ii).
If K_3 is a real intersection of $\mathcal{K}$ with the line in (iii), *then the Kiepert inscribed triangle $K_1K_2K_3$ has centroid on $\mathcal{K}$.* See Figure 11.

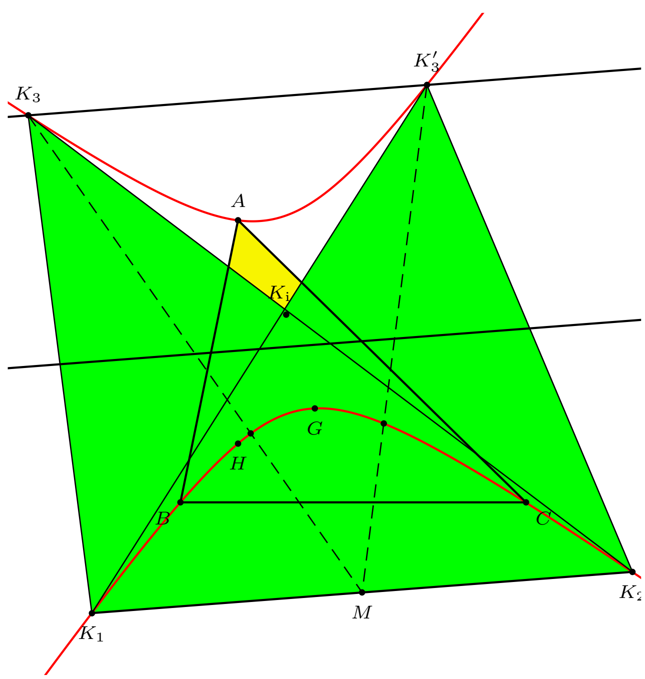

Figure 11. Construction of special Kiepert inscribed triangles given two vertices K_1, K_2

Proof. A point K_3 for which triangle $K_1K_2K_3$ has centroid on $\mathcal{K}$ clearly lies on the image of $\mathcal{K}$ under the homothety $\mathsf{h}(M, 3)$. It is therefore an intersection of $\mathcal{K}$ with this homothetic image. If $M = (u : v : w)$ in homogeneous barycentric coordinates, this homothetic conic has equation

$$\begin{aligned}&(u+v+w)^2K(x,y,z)\\+&2(x+y+z)\left(\sum((S_B-S_C)vw+(S_C-S_A)(3u+w)w+(S_A-S_B)(3u+v)v)x\right)\\=&0.\end{aligned}$$

The polar of M in $\mathcal{K}$ is the line

$$\sum((S_A-S_B)v+(S_C-S_A)w)x=0. \tag{5}$$

The parallel through M is the line

$$\sum(3(S_B-S_C)vw+(S_C-S_A)(u-w)w+(S_A-S_B)(u-v)v)x=0. \tag{6}$$

The reflection of (6) in (5) is the radical axis of $\mathcal{K}$ and its homothetic image above. $\square$

If there are two such real intersections K_3 and K_3', then the two triangles $K_1K_2K_3$ and $K_1K_2K_3'$ clearly have equal area. These two intersections coincide if the line in Construction 10 (iii) above is tangent to $\mathcal{K}$. This is the case when K_1K_2 is a tangent to the hyperbola

$$4f_2\cdot K(x,y,z)-3g_3\cdot(x+y+z)^2=0,$$

which is the image of $\mathcal{K}$ under the homothety $\mathsf{h}(K_\mathrm{i}, 2)$. See Figure 12.

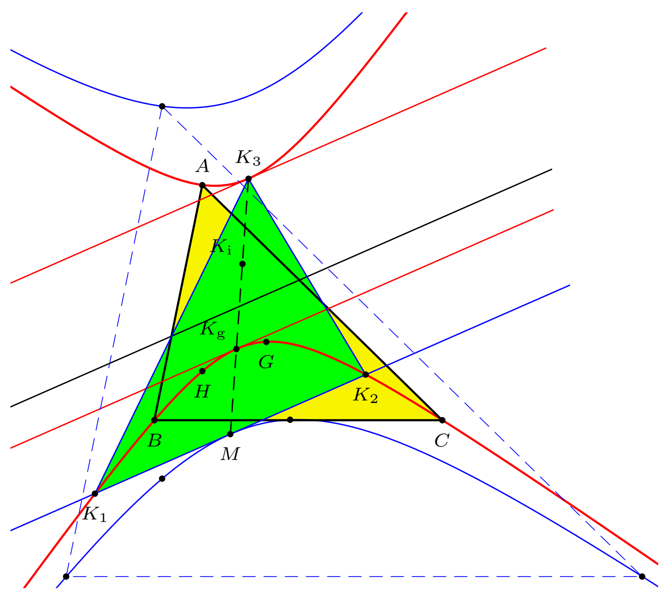

Figure 12. Family of special Kiepert inscribed triangles with K_1, K_2 uniquely determining K_3

The resulting family of special Kiepert inscribed triangles is the same family with centroid $K(t)$ and one vertex its antipode on $\mathcal{K}$, given in Construction 4.

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