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A Stronger Triangle Inequality for Neutral Geometry

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Abstract. Bailey and Bannister [*College Math. Journal*, 28 (1997) 182–186] proved that a stronger triangle inequality holds in the Euclidean plane for all triangles having largest angle less than $\arctan(\frac{24}{7}) \approx 74°$. We use hyperbolic trigonometry to show that a stronger triangle inequality holds in the hyperbolic plane for all triangles having largest angle less than or equal to $65.87°$.

1. Introduction

One of the most fundamental results of neutral geometry is the triangle inequality. How can this cherished inequality be strengthened? Under certain restrictions, the sum of the lengths of two sides of a triangle is greater than the length of the remaining side plus the length of the altitude to this side.

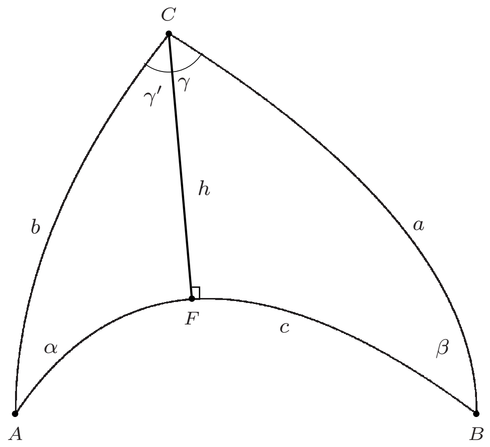

Figure 1. Strong triangle inequality $a + b > c + h$

Let ABC be a triangle belonging to neutral geometry (see Figure 1). Let a, b and c be the lengths of sides BC, AC and AB, respectively. Also, let α, β and γ denote the angles at A, B and C respectively. If we let F be the foot of the perpendicular from C onto side AB and if h is the length of the segment CF, when is it true that $a + b > c + h$? Since $a > h$ and $b > h$, this question is of interest only if c is the length of the longest side of ABC, or, equivalently, if γ is the the largest angle of ABC. With this notation, if the inequality $a + b > c + h$ holds where γ is the largest angle of the triangle ABC, we say that ABC satisfies the *strong triangle inequality*.

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The following result, due to Bailey and Bannister [1], explains what happens if the triangle ABC belongs to Euclidean geometry.

Theorem 1. *If ABC is a Euclidean triangle having largest angle $\gamma < \arctan(\frac{24}{7}) \approx 74°$, then ABC satisfies the strong triangle inequality.*

An elegant trigonometric proof of Theorem 1 can by found in [3]. It should be noted that the bound of $\arctan(\frac{24}{7})$ is the best possible since any isosceles Euclidean triangle with $\gamma = \arctan(\frac{24}{7})$ violates the strong triangle inequality.

The goal of this note is to extend the Bailey and Bannister result to neutral geometry. To get the appropriate bound for the extended result we need the function

$$f(\gamma) := -1 - \cos\gamma + \sin\gamma + \sin\frac{\gamma}{2}\sin\gamma. \tag{1}$$

Observe that $f'(\gamma) = \sin\gamma + \cos\gamma + \sin\frac{\gamma}{2}\cos\gamma + \frac{1}{2}\cos\frac{\gamma}{2}\sin\gamma > 0$ on the interval $\left[0, \frac{\pi}{2}\right]$. Therefore, $f(\gamma)$ is strictly monotone increasing on the interval $(0, \frac{\pi}{2})$. Since $f(0) = -2$, $f(\frac{\pi}{2}) = \frac{\sqrt{2}}{2}$, and f is continuous it follows that f has a unique root r in the interval $\left(0, \frac{\pi}{2}\right)$. In fact, r is approximately 1.15 (radians) or $65.87°$. See Figure 2.

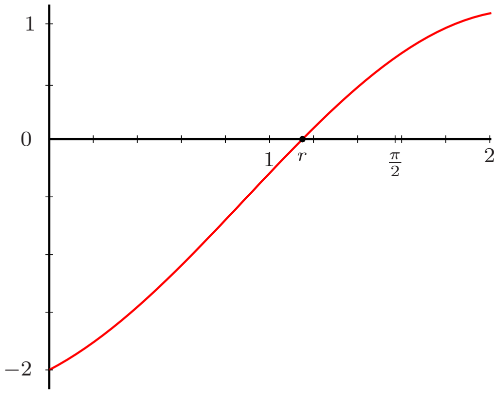

Figure 2. Graph of $f(\gamma)$

Theorem 2. *In neutral geometry a triangle ABC having largest angle γ satisfies the strong triangle inequality if $\gamma \le r \approx 1.15$ radians or $65.87°$.*

The proof of Theorem 2 is based on the fact that a model of neutral geometry is isomorphic to either the Euclidean plane or a hyperbolic plane. Given Theorem 1, it is enough to establish our result for the case of hyperbolic geometry. Moreover, since the strong triangle inequality holds if and only if $ka + kb > kc + kh$ for any positive constant k, it is enough to assume that the distance scale in hyperbolic geometry is 1. An explanation about the distance scale k and how it is used in hyperbolic geometry can be found in [4].

2. Hyperbolic trigonometry

Recall that the hyperbolic sine and hyperbolic cosine functions are given by

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

The formulas needed to prove the main result are given below. First, there are the standard identities

$$\cosh^2 x - \sinh^2 x = 1 \tag{2}$$

and

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y. \tag{3}$$

If ABC is a hyperbolic triangle with a right angle at C, *i.e.*, $\gamma = \frac{\pi}{2}$, then

$$\sinh a = \sinh c \sin \alpha \tag{4}$$

and

$$\cosh a \sin \beta = \cos \alpha. \tag{5}$$

For any hyperbolic triangle ABC,

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma, \tag{6}$$

$$\frac{\sin \alpha}{\sinh a} = \frac{\sin \beta}{\sinh b} = \frac{\sin \gamma}{\sinh c}, \tag{7}$$

$$\cosh c = \frac{\cos \alpha \cos \beta + \cos \gamma}{\sin \alpha \sin \beta}. \tag{8}$$

See [2, Chapter 10] or [5, Chapter 8] for more details regarding (4 – 8).

3. Proof of Theorem 2

The strong triangle inequality $a + b > c + h$ holds if and only if $\cosh(a+b) > \cosh(c+h)$. Expanding both sides by the identity given in (3) we have

$$\cosh a \cosh b + \sinh a \sinh b > \cosh c \cosh h + \sinh c \sinh h,$$
$$\cosh c + \sinh a \sinh b \cos \gamma + \sinh a \sinh b > \cosh c \cosh h + \sinh c \sinh h, \text{ by } (6)$$
$$\cosh c\,(1 - \cosh h) + \sinh a \sinh b\,(\cos \gamma + 1) - \sinh c \sinh h > 0.$$

Since ACF is a right triangle with the length of CF equal to h, it follows from (4) that $\sinh h = \sinh b \sin \alpha$. Applying (7), we have

$$\cosh c\,(1 - \cosh h) + \sinh a \sinh b\,(\cos \gamma + 1) - \frac{\sinh a}{\sin \alpha} \cdot \sin \gamma \sinh b \sin \alpha > 0,$$
$$\cosh c\,(1 - \cosh h) + \sinh a \sinh b\,(\cos \gamma + 1 - \sin \gamma) > 0,$$
$$\cosh c\left(1 - \cosh^2 h\right) + \sinh a \sinh b(1 + \cosh h)\,(\cos \gamma + 1 - \sin \gamma) > 0,$$
$$\cosh c\left(-\sinh^2 h\right) + \sinh a \sinh b(1 + \cosh h)\,(\cos \gamma + 1 - \sin \gamma) > 0, \quad \text{by } (2)$$
$$\cosh c\left(-\sinh^2 b \sin^2 \alpha\right) + \sinh a \sinh b(1 + \cosh h)(\cos \gamma + 1 - \sin \gamma) > 0.$$

Dividing both sides of the inequality by $\sinh b > 0$, we have

$$-\cosh c \sinh b \sin^2 \alpha + \sinh a(1 + \cosh h)(\cos \gamma + 1 - \sin \gamma) > 0.$$

By (7) and (8), we have

$$-\left(\frac{\cos \alpha \cos \beta + \cos \gamma}{\sin \alpha \sin \beta}\right) \frac{\sinh a \sin \beta}{\sin \alpha} \sin^2 \alpha + \sinh a(1+\cosh h)(\cos \gamma + 1 - \sin \gamma) > 0.$$

Simplifying and dividing by $\sinh a > 0$, we have

$$-(\cos \alpha \cos \beta + \cos \gamma) \sinh a + \sinh a(1 + \cosh h)(\cos \gamma + 1 - \sin \gamma) > 0,$$

$$-(\cos \alpha \cos \beta + \cos \gamma) + (1 + \cosh h)(\cos \gamma + 1 - \sin \gamma) > 0, \tag{9}$$

We have manipulated the original inequality into one involving the original angles, α, β, and γ, and the length of the altitude on AB. In the right triangle ACF, let γ' be the angle at C. We may assume $\gamma' \leq \frac{\gamma}{2}$ (otherwise we can work with the right triangle BCF). Applying (5) to triangle ACF gives $\cosh h = \frac{\cos \alpha}{\sin \gamma'}$. Now continuing with the inequality (9) we get

$$-(\cos \alpha \cos \beta + \cos \gamma) + \left(1 + \frac{\cos \alpha}{\sin \gamma'}\right)(1 + \cos \gamma - \sin \gamma) > 0$$

Multiplying both sides by $-\sin \gamma' < 0$, we have

$$\sin \gamma' (\cos \alpha \cos \beta + \cos \gamma) - \left(\sin \gamma' + \cos \alpha\right)(1 + \cos \gamma - \sin \gamma) < 0,$$

Simplifying this and rearranging terms, we have

$$\cos \alpha \left(\sin \gamma' \cos \beta - 1 - \cos \gamma + \sin \gamma\right) + \sin \gamma' (\sin \gamma - 1) < 0. \tag{10}$$

If $\sin \gamma' \cos \beta - 1 - \cos \alpha + \sin \alpha > 0$, then

$$\begin{aligned} &\cos \alpha \left(\sin \gamma' \cos \beta - 1 - \cos \gamma + \sin \gamma\right) + \sin \gamma' (\sin \gamma - 1) \\ &< \sin \gamma' - 1 - \cos \gamma + \sin \gamma + \sin \gamma' (\sin \gamma - 1) \\ &= -1 - \cos \gamma + \sin \gamma + \sin \gamma' \sin \gamma \\ &\leq -1 - \cos \gamma + \sin \gamma + \sin \frac{\gamma}{2} \sin \gamma. \end{aligned}$$

Note that this last expression is $f(\gamma)$ defined in (1). We have shown that

$$\cos \alpha(\sin \gamma' \cos \beta - 1 - \cos \gamma + \sin \gamma) + \sin \gamma'(\sin \gamma - 1) < \max\{0, f(\gamma)\}.$$

For $\gamma \leq r$, we have $f(\gamma) \leq 0$ and the strong triangle inequality holds.
This completes the proof of Theorem 2.

If $r < \gamma < \frac{\pi}{2}$, then $f(\gamma) > 0$. In this case, we can find an angle α such that $0 < \alpha < \frac{\pi}{2} - \frac{\gamma}{2}$ and

$$\cos \alpha \left(\sin \frac{\gamma}{2} \cos \alpha - 1 - \cos \alpha + \sin \alpha\right) + \sin \frac{\gamma}{2} (\sin \gamma - 1) > 0.$$

Since $\gamma + 2\alpha < \pi$ it follows from [5, Theorem 6.7] that there exists a hyperbolic triangle ABC with angles α, α, and γ. Our previous work shows that the triangle ABC satisfies the strong triangle inequality if and only if (10) holds. Consequently, $a + b > c + h$ provided $f(\gamma) \leq 0$. Therefore, the bound r given in Theorem 2 is the best possible.

References

[1] H. R. Bailey and R. Bannister, A stronger triangle inequality, *College Math. J.*, 28 (1997) 182–186.
[2] M. J. Greenberg, *Euclidean and Non-Euclidean Geometries, Development and History*, 3rd edition, W. H. Freeman and Company, New York, 1993.
[3] M. S. Klamkin, A sharp triangle inequality, *College Math. J.*, 29 (1998), 33.
[4] G. E. Martin, *The Foundations of Geometry and the Non-Euclidean Plane*, Undergraduate Texts in Mathematics, Springer-Verlag, New York-Berlin, 1982.
[5] S. Stahl, *The Poincaré Half-Plane, A Gateway to Modern Geometry*, Jones and Bartlett Publishers, Boston, MA, 1993.

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