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A Visual Proof of the Erdős-Mordell Inequality

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Abstract. We present a visual proof of a lemma that reduces the proof of the Erdős-Mordell inequality to elementary algebra.

In 1935, the following problem proposal appeared in the "Advanced Problems" section of the *American Mathematical Monthly* [5]:

> **3740**. *Proposed by Paul Erdős, The University, Manchester, England*.
> From a point O inside a given triangle ABC the perpendiculars OP, OQ, OR are drawn to its sides. Prove that
>
> $$OA + OB + OC \geq 2(OP + OQ + OR).$$

Trigonometric solutions by Mordell and Barrow appeared in [11]. The proofs, however, were not elementary. In fact, no "simple and elementary" proof of what had become known as the Erdős-Mordell theorem was known as late as 1956 [13]. Since then a variety of proofs have appeared, each one in some sense simpler or more elementary than the preceding ones. In 1957 Kazarinoff published a proof [7] based upon a theorem in Pappus of Alexandria's *Mathematical Collection*; and a year later Bankoff published a proof [2] using orthogonal projections and similar triangles. Proofs using area inequalities appeared in 1997 and 2004 [4, 9]. Proofs employing Ptolemy's theorem appeared in 1993 and 2001 [1, 10]. A trigonometric proof of a generalization of the inequality in 2001 [3], subsequently generalized in 2004 [6]. Many of these authors speak glowingly of this result, referring to it as a "beautiful inequality" [9], a "remarkable inequality" [12], "the famous Erdős-Mordell inequality" [4, 6, 10], and "the celebrated Erdős-Mordell inequality . . . a beautiful piece of elementary mathematics" [3].

In this short note we continue the progression towards simpler proofs. First we present a visual proof of a lemma that reduces the proof of the Erdős-Mordell inequality to elementary algebra. The lemma provides three inequalities relating the lengths of the sides of ABC and the distances from O to the vertices and to the sides. While the inequalities in the lemma are not new, we believe our proof of the lemma is. The proof uses nothing more sophisticated than elementary properties of triangles. In Figure 1(a) we see the triangle as described by Erdős, and in Figure

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1(b) we denote the lengths of relevant line segments by lower case letters, whose use will simplify the presentation to follow. In terms of that notation, the Erdős-Mordell inequality becomes

$$x + y + z \geq 2(p + q + r).$$

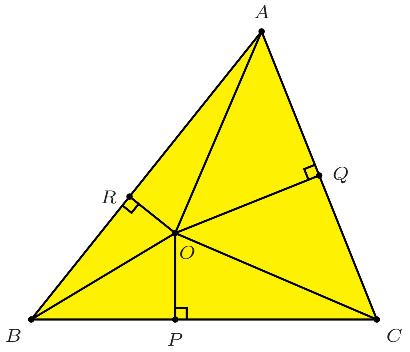

Figure 1(a)

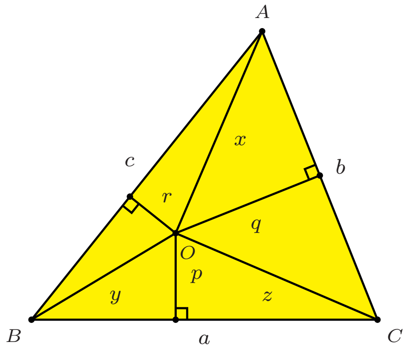

Figure 1(b)

In the proof of the lemma, we construct a trapezoid in Figure 2(b) from three triangles – one similar to ABC, the other two similar to two shaded triangles in Figure 2(a).

Lemma. *For the triangle ABC in Figure 1, we have $ax \geq br + cq$, $by \geq ar + cp$, and $cz \geq aq + bp$.*

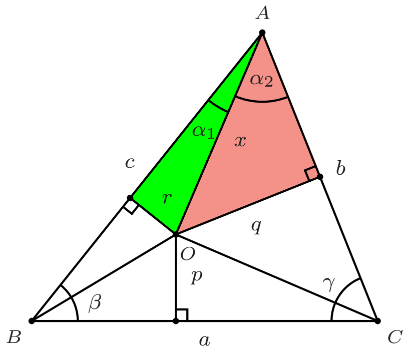

Figure 2(a)

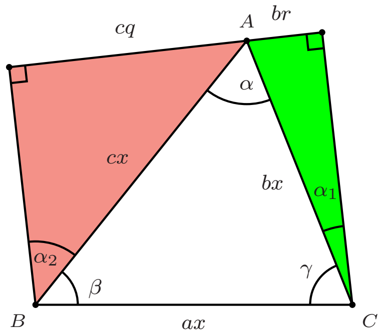

Figure 2(b)

Proof. See Figure 2 for a visual proof that $ax \geq br + cq$. The other two inequalities are established analogously. $\square$

We should note before proceeding that the object in Figure 2(b) really is a trapezoid, since the three angles at the point where the three triangles meet measure $\frac{\pi}{2} - \alpha_2$, $\alpha = \alpha_1 + \alpha_2$, and $\frac{\pi}{2} - \alpha_1$, and thus sum to π.

We now prove

The Erdős-Mordell Inequality. *If O is a point within a triangle ABC whose distances to the vertices are x, y, and z, then*

$$x + y + z \geq 2(p + q + r).$$

Proof. From the lemma we have $x \geq \frac{b}{a}r + \frac{c}{a}q$, $y \geq \frac{a}{b}r + \frac{c}{b}p$, and $z \geq \frac{a}{c}q + \frac{b}{c}p$. Adding these three inequalities yields

$$x + y + z \geq \left(\frac{b}{c} + \frac{c}{b}\right)p + \left(\frac{c}{a} + \frac{a}{c}\right)q + \left(\frac{a}{b} + \frac{b}{a}\right)r. \qquad (1)$$

But the arithmetic mean-geometric mean inequality insures that the coefficients of p, q, and r are each at least 2, from which the desired result follows. □

We conclude with several comments about the lemma and the Erdős-Mordell inequality and their relationships to other results.

1. The three inequalities in the lemma are equalities if and only if O is the center of the circumscribed circle of ABC. This follows from the observation that the trapezoid in Figure 2(b) is a rectangle if and only if $\beta + \alpha_2 = \frac{\pi}{2}$ and $\gamma + \alpha_1 = \frac{\pi}{2}$ (and similarly in the other two cases), so that $\angle AOQ = \beta = \angle COQ$. Hence the right triangles AOQ and COQ are congruent, and $x = z$. Similarly one can show that $x = y$. Hence, $x = y = z$ and O must be the circumcenter of ABC. The coefficients of p, q, and r in (1) are equal to 2 if and only if $a = b = c$. Consequently we have equality in the Erdős-Mordell inequality if and only if ABC is equilateral and O is its center.

2. How did Erdős come up with the inequality in his problem proposal? Kazarinoff [8] speculates that he generalized Euler's inequality: if $\overline{r}$ and $\overline{R}$ denote, respectively, the inradius and circumradius of ABC, then $\overline{R} \geq 2\overline{r}$. The Erdős-Mordell inequality implies Euler's inequality for acute triangles. Note that if we take O to be the circumcenter of ABC, then $3\overline{R} \geq 2(p + q + r)$. However, for *any* point O inside ABC, the quantity $p + q + r$ is somewhat surprisingly constant and equal to $\overline{R} + \overline{r}$, a result known as Carnot's theorem. Thus $3\overline{R} \geq 2(\overline{R} + \overline{r})$, or equivalently, $\overline{R} \geq 2\overline{r}$.

3. Many other inequalities relating x, y, and z to p, q, and r can be derived. For example, applying the arithmetic mean-geometric mean inequality to the right side of the inequalities in the lemma yields

$$ax \geq 2\sqrt{bcqr}, \quad by \geq 2\sqrt{carp}, \quad cz \geq 2\sqrt{abpq}.$$

Multiplying these three inequalities together and simplifying yields $xyz \geq 8pqr$. More such inequalities can be found in [8, 12].

4. A different proof of (1) appears in [4].

References

[1] A. Avez, A short proof of a theorem of Erdős and Mordell, *Amer. Math. Monthly*, 100 (1993) 60–62.

[2] L. Bankoff, An elementary proof of the Erdős-Mordell theorem, *Amer. Math. Monthly*, 65 (1958) 521.

[3] S. Dar and S. Gueron, A weighted Erdős-Mordell inequality, *Amer. Math. Monthly*, 108 (2001) 165–168.
[4] N. Dergiades, Signed distances and the Erdős-Mordell inequality, *Forum Geom.*, 4 (2004) 67–68.
[5] P. Erdős, Problem 3740, *Amer. Math. Monthly*, 42 (1935) 396.
[6] W. Janous, Further inequalities of Erdős-Mordell type, *Forum Geom.*, 4 (2004) 203–206.
[7] D. K. Kazarinoff, A simple proof of the Erdős-Mordell inequality for triangles, *Michigan Math. J.*, 4 (1957) 97–98.
[8] N. D. Kazarinoff, *Geometric Inequalities*, MAA, Washington, 1961.
[9] V. Komornik, A short proof of the Erdős-Mordell theorem, *Amer. Math. Monthly*, 104 (1997) 57–60.
[10] H. Lee, Another proof of the Erdős-Mordell theorem, *Forum Geom.*, 1 (2001) 7–8.
[11] L. J. Mordell and D. F. Barrow, Solution 3740, *Amer. Math. Monthly*, 44 (1937) 252–254.
[12] A. Oppenheim, The Erdős inequality and other inequalities for a triangle, *Amer. Math. Monthly*, 68 (1961) 226–230.
[13] G. Steensholt, Note on an elementary property of triangles, *Amer. Math. Monthly*, 63 (1956) 571–572.

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