Forum Geometricorum
Volume 7 (2007) 103–106.

FORUM GEOM
ISSN 1534-1178

Construction of Triangle from a Vertex and the Feet of Two Angle Bisectors

Harold Connelly, Nikolaos Dergiades, and Jean-Pierre Ehrmann

Abstract. We give two simple constructions of a triangle given one vertex and the feet of two angle bisectors.

1. Construction from $(A,\ T_a,\ T_b)$

We present two simple solutions of the following construction problem (number 58) in the list compiled by W. Wernick [2]: Given three noncollinear points A, T_a and T_b, to construct a triangle ABC with T_a, T_b on BC, CA respectively such that AT_a and BT_b are bisectors of the triangle. L. E. Meyers [1] has indicated the constructibility of such a triangle. Let ℓ be the half line AT_b. Both solutions we present here make use of the reflection ℓ' of ℓ in AT_a. The vertex B necessarily lies on ℓ'. In what follows $P(Q)$ denotes the circle, center P, passing through the point Q.

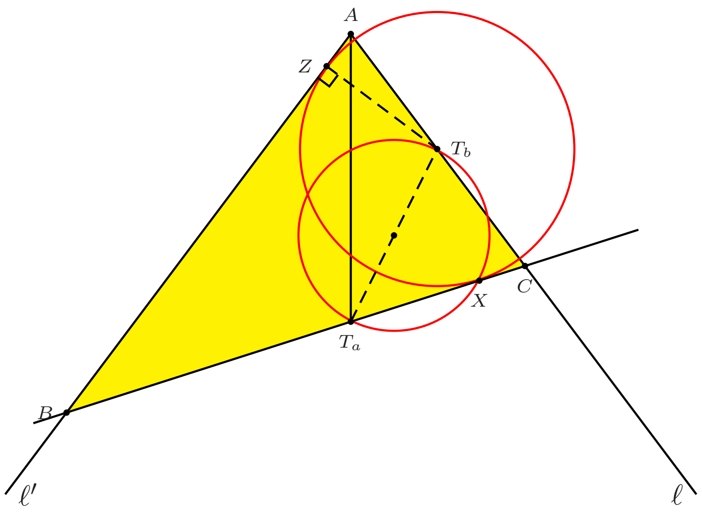

Figure 1

Construction 1. *Let Z be the pedal of T_b on ℓ'. Construct two circles, one $T_b(Z)$, and the other with T_aT_b as diameter. Let X be an intersection, if any, of these two circles. If the line XT_a intersects the half lines ℓ' at B and ℓ at C, then ABC is a desired triangle.*

Publication Date: May 7, 2007. Communicating Editor: Paul Yiu.

Construction 2. *Let P be an intersection, if any, of the circle $T_b(T_a)$ with the half line ℓ'. Construct the perpendicular bisector of PT_a. If this intersects ℓ' at a point B, and if the half line BT_a intersects ℓ at C, then ABC is a desired triangle.*

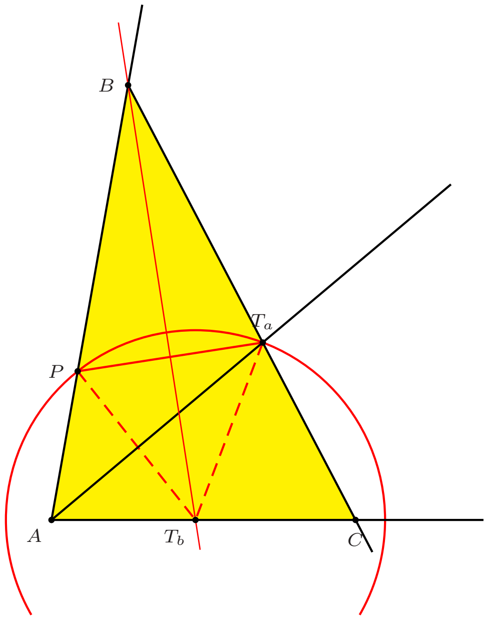

Figure 2

We study the number of solutions for various relative positions of A, T_a and T_b. Set up a polar coordinate system with A at the pole and T_b at $(1, 0)$. Suppose T_a has polar coordinates $(\rho,\ \theta)$ for $\rho > 0$ and $0 < \theta < \frac{\pi}{2}$. The half line ℓ' has polar angle 2θ. The circle $T_b(T_a)$ intersects ℓ' if the equation

$$\sigma^2 - 2\sigma\cos 2\theta = \rho^2 - 2\rho\cos\theta \tag{1}$$

has a positive root σ. This is the case when
(i) $\rho > 2\cos\theta$, or
(ii) $\rho \leq 2\cos\theta$, $\cos 2\theta > 0$ and $4\cos^2 2\theta + 4\rho(\rho - 2\cos\theta) \geq 0$. Equivalently, $\rho_+ \leq \rho \leq 2\cos\theta$, where

$$\rho_\pm = \cos\theta \pm \sqrt{\sin\theta\sin 3\theta}$$

are the roots of the equation $\rho^2 - 2\rho\cos\theta + \cos^2 2\theta = 0$ for $0 < \theta < \frac{\pi}{3}$.

Now, the perpendicular bisector of PT_a intersects the line ℓ' at the point B with polar coordinates $(\beta,\ 2\theta)$, where

$$\beta = \frac{\rho\cos\theta - \sigma\cos 2\theta}{\rho\cos\theta - \sigma}.$$

The requirement $\beta > 0$ is equivalent to $\sigma < \rho\cos\theta$. From (1), this is equivalent to $\rho < 4\cos\theta$.

For $0 < \theta < \frac{\pi}{3}$, let $P_\pm$ be the points with polar coordinates $(\rho_\pm,\ \theta)$. These points bound a closed curve $\mathscr{C}$ as shown in Figure 3. If T_a lies inside the curve $\mathscr{C}$, then the circle $T_b(T_a)$ does not intersect the half line ℓ'. We summarize the results with reference to Figure 3.

The construction problem of ABC from $(A,\ T_a,\ T_b)$ has
(1) a unique solution if T_a lies in the region between the two semicircles $\rho = 2\cos\theta$ and $\rho = 4\cos\theta$,
(2) two solutions if T_a lies between the semicircle $\rho = 2\cos\theta$ and the curve $\mathscr{C}$ for $\theta < \frac{\pi}{4}$.

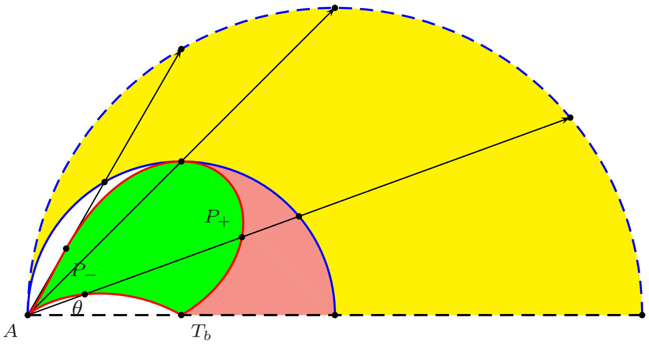

Figure 3.

2. Construction from $(A,\ T_b,\ T_c)$

The construction of triangle ABC from $(A,\ T_a,\ T_b)$ is Problem 60 in Wernick's list [2]. Wernick has indicated constructibility. We present two simple solutions.

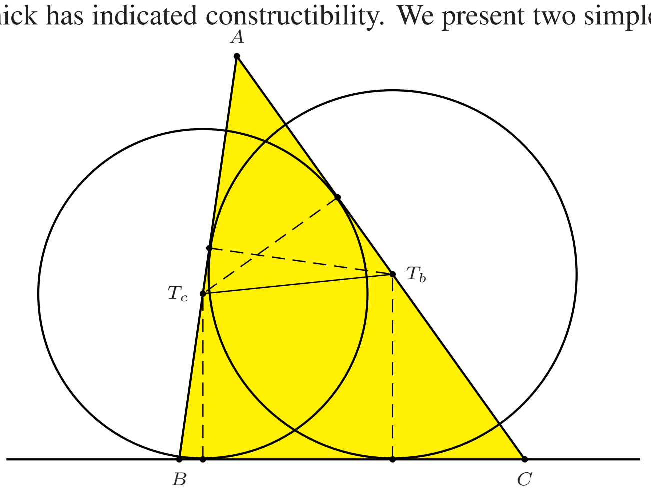

Figure 4.

Construction 3. *Given A, T_b, T_c, construct the circles with centers T_b and T_b, tangent to AT_c and AT_b respectively. The common tangent of these circles that lies opposite to A with respect to the line T_bT_c is the line BC of the required triangle ABC. The construction of the vertices B, C is obvious.*

Construction 4. *Given A, T_b, T_c, construct*
(i) *the circle through the three points*
(ii) *the bisector of angle T_bAT_c to intersect the circle at M,*
(iii) *the reflection M' of M in the line T_bT_c,*
(iv) *the circle $M'(T_b)$ to intersect the bisector at I (so that A and I are on opposite sides of T_bT_c),*
(v) *the half line T_bI to intersect the half line AT_c at B,*
(vi) *the half line T_cI to intersect the half line AT_b at C.*
ABC is the required triangle with incenter I.

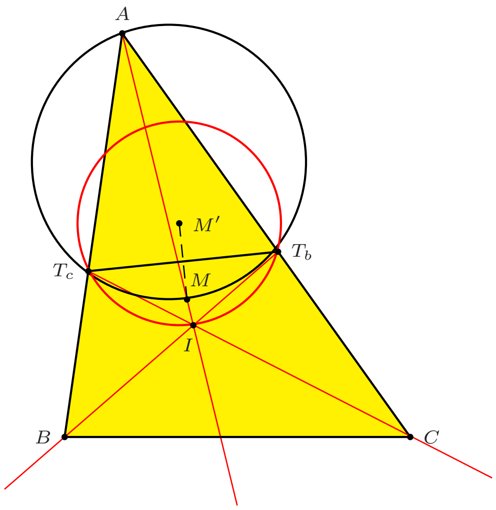

Figure 5.

References

[1] L. E. Meyers, Update on William Wernick's "triangle constructions with three located points", *Math. Mag.*, 69 (1996) 46–49.
[2] W. Wernick, Triangle constructions with three located points, *Math. Mag.*, 55 (1982) 227–230.

Harold Connelly: 102 Blowing Tree Drive, Georgetown, Kentucky 40324, USA
E-mail address: cherylandharold@roadrunner.com

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: ndergiades@yahoo.gr

Jean-Pierre Ehrmann: 6, rue des Cailloux, 92110 - Clichy, France
E-mail address: Jean-Pierre.EHRMANN@wanadoo.fr