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Three Pappus Chains Inside the Arbelos: Some Identities

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Abstract. We consider the three different Pappus chains that can be constructed inside the arbelos and we deduce some identities involving the radii of the circles of n-th order and the incircle radius.

1. Introduction

The Pappus chain [1] is an infinite series of circles constructed starting from the Archimedean figure named arbelos (also said shoemaker knife) so that the generic circle $\mathcal{C}_i$, $(i = 1,\ 2,\ \ldots)$ of the chain is tangent to the the circles $\mathcal{C}_{i-1}$ and $\mathcal{C}_{i+1}$ and to two of the three semicircles $\mathcal{C}_{\rm a}$, $\mathcal{C}_{\rm b}$ and $\mathcal{C}_{\rm r}$ forming the arbelos. In a generic arbelos three different Pappus chains can be drawn (see Figure 1).

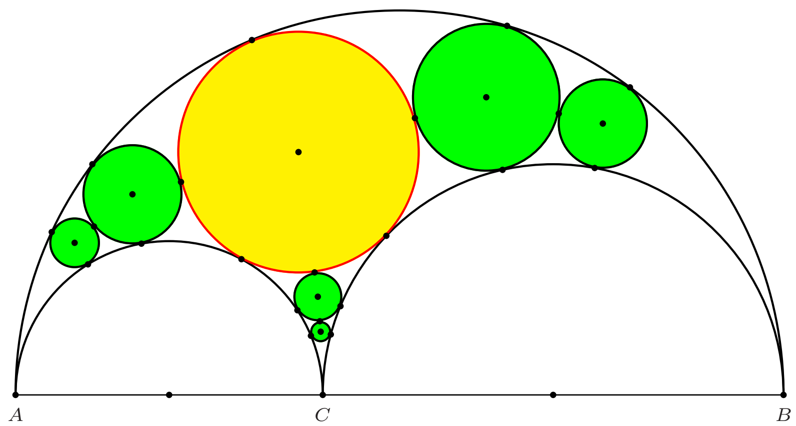

Figure 1.

In Figure 1, the diameter AC of the left semicircle $\mathcal{C}_{\rm a}$ is $2a$, the diameter CB of the right semicircle $\mathcal{C}_{\rm b}$ is $2b$, and the diameter AB of the outer semicircle $\mathcal{C}_{\rm r}$ is $2r$, $r = a + b$. The first circle $\mathcal{C}_1$ is common to all three chains and is named the incircle of the arbelos. By applying the circular inversion technique, it is possible to determine the center coordinates and radius of each chain; the radii are expressed by the formulas reported in Table I. The chain tending to point C is named $\Gamma_{\rm r}$, the chain tending to point B is named $\Gamma_{\rm a}$ and the chain tending to point A is named $\Gamma_{\rm b}$. As far as chains $\Gamma_{\rm a}$ and $\Gamma_{\rm b}$ are concerned, the expressions for the radii are given in [2] while for $\Gamma_{\rm r}$, we give an inductive proof below.

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Table I: Radii of the circles forming the three Pappus chains

Chain	$\Gamma_{\rm r}$	$\Gamma_{\rm a}$	$\Gamma_{\rm b}$
Radius of n-th circle	$\rho_{\rm rn} = \frac{rab}{n^2r^2-ab}$	$\rho_{\rm an} = \frac{rab}{n^2a^2+rb}$	$\rho_{\rm bn} = \frac{rab}{n^2b^2+ra}$

For integers $n \geq 1$, consider the statement

$$P(n) \qquad \rho_{\rm rn} = \frac{rab}{n^2r^2 - ab}.$$

$P(1)$ is true since the first circle of the chain is the arbelos incircle having radius given by formula (3).

We show that $P(n) \Rightarrow P(n+1)$.

Let us consider the circles $\mathcal{C}_{\rm rn}$ and $\mathcal{C}_{\rm rn+1}$ in the chain $\Gamma_{\rm r}$, together the inner semicircles $\mathcal{C}_{\rm a}$ and $\mathcal{C}_{\rm b}$ inside the arbelos. Applying Descartes' theorem we have

$$2(\varepsilon_{{\rm r}n}^2 + \varepsilon_{{\rm r}n+1}^2 + \varepsilon_{\rm a}^2 + \varepsilon_{\rm b}^2) = (\varepsilon_{{\rm r}n} + \varepsilon_{{\rm r}n+1} + \varepsilon_{\rm a} + \varepsilon_{\rm b})^2, \tag{1}$$

where $\varepsilon_{{\rm r}n}$, $\varepsilon_{{\rm r}n+1}$, $\varepsilon_{\rm a}$ and $\varepsilon_{\rm b}$ are the curvatures, *i.e.*, reciprocals of the radii of the circles. Rewriting this as

$$\varepsilon_{{\rm r}n+1}^2 - 2\varepsilon_{{\rm r}n+1}(\varepsilon_{{\rm r}n} + \varepsilon_{\rm a} + \varepsilon_{\rm b}) + \varepsilon_{{\rm r}n}^2 + \varepsilon_{\rm a}^2 + \varepsilon_{\rm b}^2 - 2(\varepsilon_{{\rm r}n}\varepsilon_{\rm a} + \varepsilon_{\rm a}\varepsilon_{\rm b} + \varepsilon_{\rm b}\varepsilon_{{\rm r}n}) = 0,$$

we have

$$\varepsilon_{{\rm r}n+1} = \varepsilon_{{\rm r}n} + \varepsilon_{\rm a} + \varepsilon_{\rm b} \pm 2\sqrt{\varepsilon_{{\rm r}n}\varepsilon_{\rm a} + \varepsilon_{\rm a}\varepsilon_{\rm b} + \varepsilon_{\rm b}\varepsilon_{{\rm r}n}}. \tag{2}$$

Substituting into (2) $\varepsilon_{\rm a} = \frac{1}{a}$, $\varepsilon_{\rm b} = \frac{1}{b}$ and $\varepsilon_{\rm rn} = \frac{rab}{n^2r^2-ab}$, we obtain, after a few steps of simple algebraic calculations,

$$\rho_{{\rm r}n+1} = \frac{1}{\varepsilon_{{\rm r}n+1}} = \frac{rab}{(n+1)^2r^2 - ab}.$$

This proves that $P(n) \Rightarrow P(n+1)$, and by induction, $P(n)$ is true for every integer $n \geq 1$.

2. Relationships among the n-th circles radii and incircle radius

For the following, it is useful to write explicitly the incircle radius $\rho_{\rm inc}$ that is given by:

$$\rho_{\rm inc} = \frac{rab}{a^2 + ab + b^2} \tag{3}$$

Formula (3) is directly obtained by each one of the three formulas for the radius in Table I for $n = 1$. It is useful too to write the square of the incircle radius that is:

$$\rho_{\rm inc}^2 = \frac{r^2a^2b^2}{a^4 + 2a^3b + 2a^2b^2 + 2ab^3 + b^4}. \tag{4}$$

We enunciate now the following proposition related to three different identities among the circles chains radii and the incircle radius.

Proposition. *Given a generic arbelos with its three Pappus chains, the following identities hold for each integer n:*

$$\rho_{\mathrm{inc}}\left(\frac{1}{\rho_{\mathrm{r}n}}+\frac{1}{\rho_{\mathrm{a}n}}+\frac{1}{\rho_{\mathrm{b}n}}\right)=2n^2+1, \tag{5}$$

$$\rho_{\mathrm{inc}}^2\left(\frac{1}{\rho_{\mathrm{r}n}^2}+\frac{1}{\rho_{\mathrm{a}n}^2}+\frac{1}{\rho_{\mathrm{b}n}^2}\right)=2n^4+1, \tag{6}$$

$$\rho_{\mathrm{inc}}^2\left(\frac{1}{\rho_{\mathrm{r}n}}\cdot\frac{1}{\rho_{\mathrm{a}n}}+\frac{1}{\rho_{\mathrm{a}n}}\cdot\frac{1}{\rho_{\mathrm{b}n}}+\frac{1}{\rho_{\mathrm{b}n}}\cdot\frac{1}{\rho_{\mathrm{r}n}}\right)=n^4+2n^2. \tag{7}$$

Proof. To demonstrate (5), one has to substitute in it the expression for the radius incircle given by (3) and the expressions for the radii of n-th circles chain given in Table I. Using the fact that $r=a+b$, one obtains

$$\frac{rab}{a^2+ab+b^2}\left(\frac{n^2r^2-ab}{rab}+\frac{n^2a^2+rb}{rab}+\frac{n^2b^2+ra}{rab}\right)=2n^2+1.$$

For (6), one has to substitute in it the expression for the square of the radius incircle given by (4) and to take the squares of the radii of n-th circles chain given in Table I. Using the fact that $r=a+b$, one obtains

$$\frac{r^2a^2b^2}{(a^2+ab+b^2)^2}\left(\left(\frac{n^2r^2-ab}{rab}\right)^2+\left(\frac{n^2a^2+rb}{rab}\right)^2+\left(\frac{n^2b^2+ra}{rab}\right)^2\right)=2n^4+1.$$

For (7), one has to substitute in it the expression for the square of the incircle radius given by (4) and the expressions for the radii of the n-th circles given in Table I. This leads to $\frac{r^2a^2b^2}{(a^2+ab+b^2)^2}\cdot\frac{D}{r^2a^2b^2}$, where

$$\begin{aligned}D=&(n^2r^2-ab)(n^2a^2+rb)+(n^2a^2+rb)(n^2b^2+ra)+(n^2b^2+ra)(n^2r^2-ab)\\=&(n^4+2n^2)(a^2+ab+b^2)^2,\end{aligned}$$

by using the fact that $r=a+b$. Finally, this leads to (7). □

3. Conclusion

Considering the three Pappus chains that can be drawn inside a generic arbelos, some identities involving the incircle radius and the n-th circles chain radii have been shown. All these identities generate sequences of integers.

References

[1] F. M. van Lamoen and E. W. Weisstein, Pappus Chain, MathWorld-A Wolfram Web Resource, http://mathworld.wolfram.com/PappusChain.html
[2] L. Bankoff, The golden arbelos, *Scripta Math.*, 21 (1955) 70–76.

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