Forum Geometricorum
Volume 7 (2007) 147–149.

FORUM GEOM
ISSN 1534-1178

On the Diagonals of a Cyclic Quadrilateral

Claudi Alsina and Roger B. Nelsen

Abstract. We present visual proofs of two lemmas that reduce the proofs of expressions for the lengths of the diagonals and the area of a cyclic quadrilateral in terms of the lengths of its sides to elementary algebra.

The purpose of this short note is to give a new proof of the following well-known results of Brahmagupta and Parameśhvara [4, 5].

Theorem. If a, b, c, d denote the lengths of the sides; p, q the lengths of the diagonals, R the circumradius, and Q the area of a cyclic quadrilateral, then

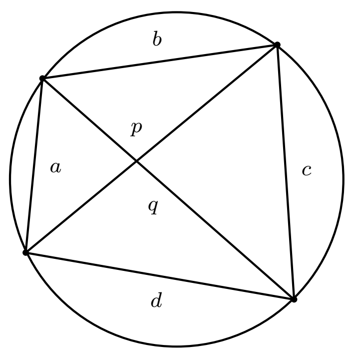

Figure 1

$$p=\sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}, \qquad q=\sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}},$$

and

$$Q=\frac{1}{4R}\sqrt{(ab+cd)(ac+bd)(ad+bc)}.$$

We begin with visual proofs of two lemmas, which will reduce the proof of the theorem to elementary algebra. Lemma 1 is the well-known relationship for the area of a triangle in terms of its circumradius and three side lengths; and Lemma 2 expresses the ratio of the diagonals of a cyclic quadrilateral in terms of the lengths of the sides.

Lemma 1. *If a, b, c denote the lengths of the sides, R the circumradius, and K the area of a triangle, then $K=\frac{abc}{4R}$.*

Publication Date: September 24, 2007. Communicating Editor: Paul Yiu.

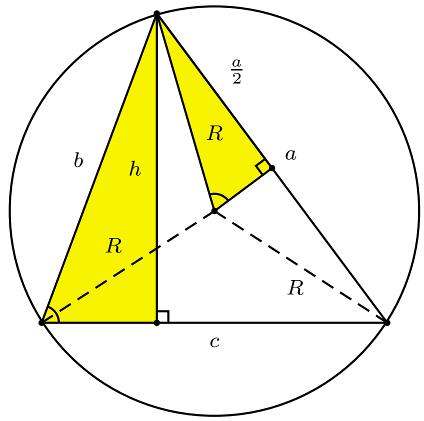

Figure 2

Proof. From Figure 2,

$$\frac{h}{b} = \frac{\frac{a}{2}}{R} \Rightarrow h = \frac{ab}{2R} \Rightarrow K = \frac{1}{2}hc = \frac{abc}{4R}.$$

□

Lemma 2 ([2]). *Under the hypotheses of the Theorem,* $\frac{p}{q} = \frac{ad+bc}{ab+cd}$.

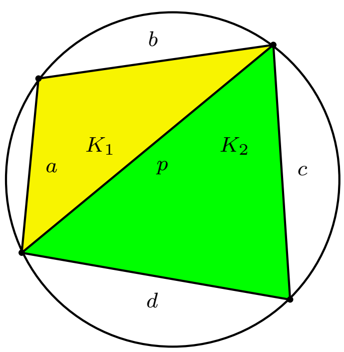

Figure 3

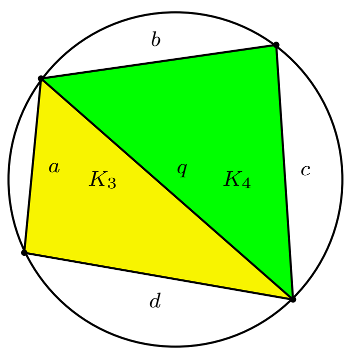

Figure 4

Proof. From Figures 3 and 4 respectively,

$$Q = K_1 + K_2 = \frac{pab}{4R} + \frac{pcd}{4R} = \frac{p(ab+cd)}{4R},$$
$$Q = K_3 + K_4 = \frac{qad}{4R} + \frac{qbc}{4R} = \frac{q(ad+bc)}{4R}.$$

Therefore,

$$\begin{aligned} p(ab+cd) =& q(ad+bc), \\ \frac{p}{q} =& \frac{ad+bc}{ab+cd}. \end{aligned}$$

□

In the proof of our theorem, we use Lemma 2 and Ptolemy's theorem: Under the hypotheses of our theorem,

$$pq = ac + bd.$$

For proofs of Ptolemy's theorem, see [1, 3].

Proof of the Theorem.

$$\begin{aligned} p^2 =& pq \cdot \frac{p}{q} = \frac{(ac+bd)(ad+bc)}{ab+cd}, \\ q^2 =& pq \cdot \frac{q}{p} = \frac{(ac+bd)(ab+cd)}{ad+bc}; \\ Q^2 =& \frac{pq(ab+cd)(ad+bc)}{(4R)^2} = \frac{(ac+bd)(ab+cd)(ad+bc)}{(4R)^2}. \end{aligned}$$

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Claudi Alsina: Secció de Matemàtiques, ETSAB, Universitat Politècnica de Catalunya, E-08028 Barcelona, Spain

E-mail address: `claudio.alsina@upc.edu`

Roger B. Nelsen: Department of Mathematical Sciences, Lewis & Clark College, Portland, Oregon 97219, USA

E-mail address: `nelsen@lclark.edu`