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On a Product of Two Points Induced by Their Cevian Triangles

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Abstract. The intersections of the corresponding sidelines of the cevian triangles of two points P_0 and P_1 form the anticevian triangle of a point $T(P_0, P_1)$. We prove a number of interesting results relating the pair of inscribed conics with perspectors (Brianchon points) P_0 and P_1, in particular, a simple description of the fourth common tangent of the conics. We also show that the corresponding sides of the cevian triangles of points are concurrent if and only if the points lie on a circumconic. A characterization is given of circumconics whose centers lie on the cevian circumcircles of points on them (Brianchon-Poncelet theorem). We also construct a number of new triangle centers with very simple coordinates.

1. Introduction

A famous problem in triangle geometry [8] asks to show that the corresponding sidelines of the orthic triangle, the intouch triangle, and the cevian triangle of the incenter are concurrent.

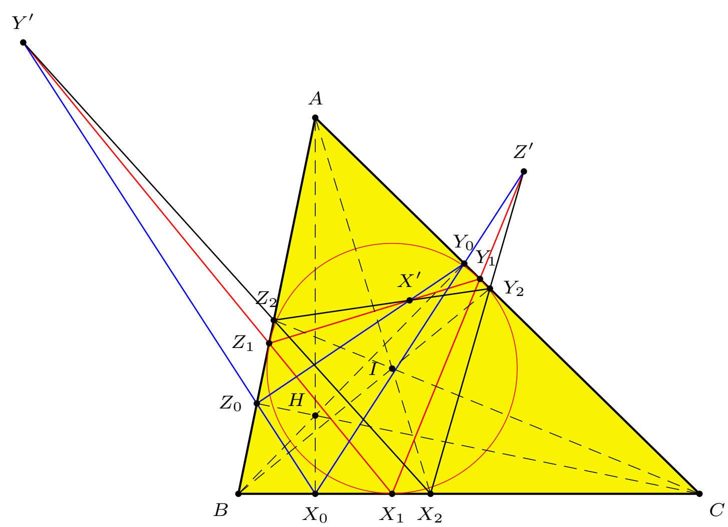

Figure 1.

Given a triangle ABC with orthic triangle $X_0Y_0Z_0$ and intouch triangle $X_1Y_1Z_1$, let

$$X' = Y_0Z_0 \cap Y_1Z_1, \qquad Y' = Z_0X_0 \cap Z_1X_1, \qquad Z' = X_0Y_0 \cap X_1Y_1.$$

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Emelyanov and Emelyanova [2] have proved the following interesting theorem. If XYZ is an inscribed triangle (with X, Y, Z on the sidelines BC, CA, AB respectively, and Y' on XZ and Z' on XY), then the circle through X, Y, Z also passes through the Feuerbach point, the point of tangency of the incircle with the nine-point circle of triangle ABC.

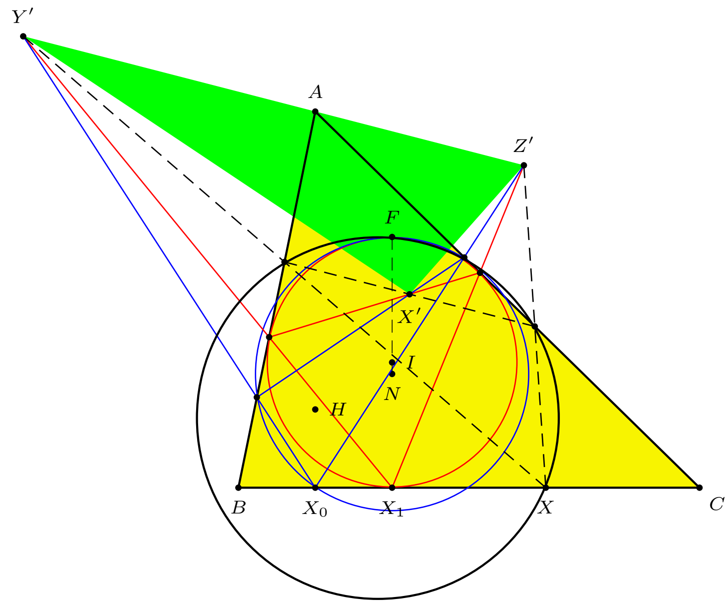

Figure 2.

In this note we study a general situation which reveals more of the nature of these theorems. By showing that the intersections of the corresponding sidelines of the cevian triangles of two points P_0 and P_1 form the anticevian triangle of a point $T(P_0, P_1)$, we prove a number of interesting results relating the pair of inscribed conics with perspectors (Brianchon points) P_0 and P_1. Proposition 5 below shows that the corresponding sidelines of the cevian triangles of three points are concurrent if and only if the three points lie on a circumconic. We characterize such circumconics whose centers lie on the cevian circumcircles of points on them (Proposition 9).

We shall work with homogeneous barycentric coordinates with reference to triangle ABC, and make use of standard notations of triangle geometry. A basic reference is [10]. Except for the commonest ones, triangle centers are labeled according to [7].

2. A product induced by two cevian triangles

Let $P_0 = (u_0 : v_0 : w_0)$ and $P_1 = (u_1 : v_1 : w_1)$ be two given points, with cevian triangles $X_0Y_0Z_0$ and $X_1Y_1Z_1$ respectively. The intersections

$$X' = Y_0Z_0 \cap Y_1Z_1, \qquad Y' = Z_0X_0 \cap Z_1X_1, \qquad Z' = X_0Y_0 \cap X_1Y_1$$

are the vertices of the anticevian triangle of a point with homogeneous barycentric coordinates

$$\left(u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right):\ v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right):\ w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)\right) \tag{1}$$

$$=\left(u_1\left(\frac{v_1}{v_0}-\frac{w_1}{w_0}\right):\ v_1\left(\frac{w_1}{w_0}-\frac{u_1}{u_0}\right):\ w_1\left(\frac{u_1}{u_0}-\frac{v_1}{v_0}\right)\right). \tag{2}$$

That these two sets of coordinates should represent the same point is quite clear geometrically. They define a product of P_0 and P_1 which clearly lies on the trilinear polars of P_0 and P_1. This product is therefore the intersection of the trilinear polars of P_0 and P_1. We denote this product by $T(P_0,\ P_1)$.

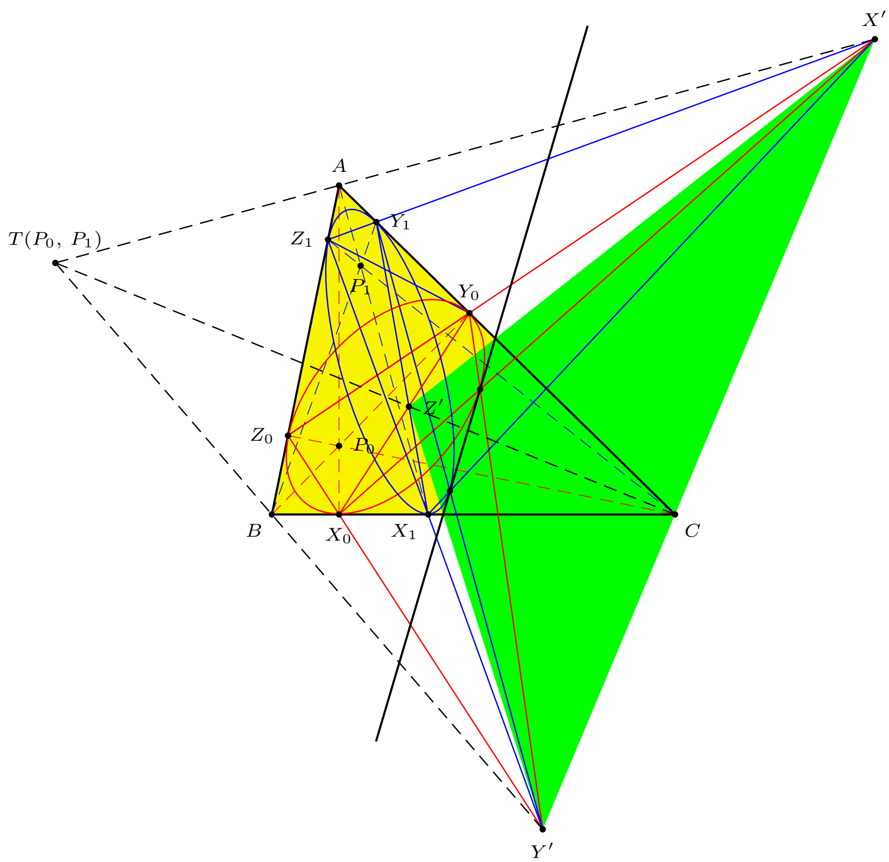

Figure 3.

The point $T(P_0,\ P_1)$ is also the perspector of the circumconic through P_0 and P_1. In particular, if P_0 and P_1 are both on the circumcircle, then $T(P_0,\ P_1)=K$, the symmedian point.

Proposition 1. *Triangle $X'Y'Z'$ is perspective to*
(i) *triangle $X_0Y_0Z_0$ at the point*

$$P_0/(T(P_0,\ P_1)) = \left(u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)^2 : v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)^2 : w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)^2\right),$$

(ii) *triangle $X_1Y_1Z_1$ at the point*

$$P_1/(T(P_0,\ P_1)) = \left(u_1\left(\frac{v_1}{v_0}-\frac{w_1}{w_0}\right)^2 : v_1\left(\frac{w_1}{w_0}-\frac{u_1}{u_0}\right)^2 : w_1\left(\frac{u_1}{u_0}-\frac{v_1}{v_0}\right)^2\right).$$

Proof. Since $X'Y'Z'$ is an anticevian triangle, the perspectivity is clear in each case by the cevian nest theorem (see [10, §8.3] and [4, p.165, Supp. Exercise 7]). The perspectors are the cevian quotients $P_0/(T(P_0,\ P_1))$ and $P_1/(T(P_0,\ P_1))$. We need only consider the first case.

$$\begin{aligned}
&P_0/(T(P_0,\ P_1))\\
=&\left(u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)\left(-\frac{u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)}{u_0}+\frac{v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)}{v_0}+\frac{w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)}{w_0}\right)\right.\\
&\quad : v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)\left(\frac{u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)}{u_0}-\frac{v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)}{v_0}+\frac{w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)}{w_0}\right)\\
&\quad \left. : w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)\left(\frac{u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)}{u_0}+\frac{v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)}{v_0}-\frac{w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)}{w_0}\right)\right)\\
=&\left(u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)^2 :\ v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)^2 :\ w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)^2\right).
\end{aligned}$$

□

Remark. See Proposition 12 for another triangle whose sidelines contain the points X', Y', Z'.

The conic with perspector P_0 has equation

$$\frac{x^2}{u_0^2}+\frac{y^2}{v_0^2}+\frac{z^2}{w_0^2}-\frac{2yz}{v_0w_0}-\frac{2zx}{w_0u_0}-\frac{2xy}{u_0v_0}=0,$$

and each point on the conic is of the form $(u_0p^2 : v_0q^2 : w_0r^2)$ for $p+q+r=0$. From this it is clear that $P_0/(T(P_0,\ P_1))$ lies on the inscribed conic with perspector P_0. Similarly, $P_1/(T(P_0,\ P_1))$ lies on that with perspector P_1.

Proposition 2. *The line joining $P_0/(T(P_0,\ P_1))$ and $P_1/(T(P_0,\ P_1))$ is the trilinear polar of $T(P_0,\ P_1)$ with respect to triangle ABC. It is also the (fourth) common tangent of the two inscribed conics with perspectors P_0 and P_1.* (See Figure 3).

Proof. For the first part it is enough to verify that $P_0/(T(P_0,\ P_1))$ lies on the said trilinear polar:

$$\frac{u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)^2}{u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)}+\frac{v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)^2}{v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)}+\frac{w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)^2}{w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)}=0.$$

Note that the coordinates of $T(P_0,\ P_1)$ are given by both (1) and (2). Interchanging the subscripts 0's and 1's shows that the trilinear polar of $T(P_0,\ P_1)$ also contains the point $P_1/(T(P_0,\ P_1))$.

The inscribed conic with perspector P_0 is represented by the matrix

$$\begin{pmatrix} \frac{1}{u_0^2} & \frac{-1}{u_0v_0} & \frac{-1}{u_0w_0} \\ \frac{-1}{u_0v_0} & \frac{1}{v_0^2} & \frac{-1}{v_0w_0} \\ \frac{-1}{u_0w_0} & \frac{-1}{v_0w_0} & \frac{1}{w_0^2} \end{pmatrix}.$$

The tangent at the point $P_0/(T(P_0,\ P_1))$ has line coordinates

$$\begin{pmatrix} \frac{1}{u_0^2} & \frac{-1}{u_0v_0} & \frac{-1}{u_0w_0} \\ \frac{-1}{u_0v_0} & \frac{1}{v_0^2} & \frac{-1}{v_0w_0} \\ \frac{-1}{u_0w_0} & \frac{-1}{v_0w_0} & \frac{1}{w_0^2} \end{pmatrix}\begin{pmatrix} u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)^2 \\ v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)^2 \\ w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)^2 \end{pmatrix}=\begin{pmatrix} \frac{2}{u_0}\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right) \\ \frac{2}{v_0}\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right) \\ \frac{2}{w_0}\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right). \end{pmatrix}$$

This is the line

$$\frac{x}{u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)}+\frac{y}{v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)}+\frac{z}{w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)}=0,$$

which is the trilinear polar of $T(P_0,\ P_1)$. Interchanging the subscripts 0's and 1's, we note that the same line is also the tangent at the point $P_1/(T(P_0,\ P_1))$of the inscribed conics with perspector P_1. It is therefore the common tangent of the two conics. $\square$

Proposition 3. *The triangle $X'Y'Z'$ is self polar with respect to each of the inscribed conics with perspectors P_0 and P_1.*

Proof. Since $X_1Y_1Z_1$ is a cevian triangle and $X'Y'Z'$ is an anticevian triangle with respect to ABC, we have

$$(Y'Z_0, Y'A, Y'Y_0, Y'C) = (Y'Z_0, Y'A, Y'Y_0, Y'X') = -1.$$

Therefore, Y' lies on the polar of X' with respect to the inscribed conic with perspector P_0. Similarly, Z' also lies on the polar of X'. It follows that $Y'Z'$ is the polar of X'. For the same reason, $Z'X'$ and $X'Y'$ are the polars of Y' and Z' respectively. This shows that triangle $X'Y'Z'$ is self-polar with respect to the inscribed conic with perspector P_0. The same is true with respect to the inscribed conic with perspector P_1. $\square$

In the case of the incircle (with $P_0 = X_7$), we have the following interesting result.

Corollary 4. *For an arbitrary point Q, the anticevian triangle $X_7 * Q$ has orthocenter I.*

We present some examples of $T(P_0,\ P_1)$.

	G	O	H	K	$G\mathrm{e}$	N_a	E
I	X_{513}	X_{652}	X_{650}	X_{649}	X_{650}	X_{650}	X_{2245}
G		X_{520}	X_{523}	X_{512}	X_{514}	X_{522}	X_{511}
O			X_{647}	X_{647}			
H				X_{647}	X_{650}	X_{650}	X_{3003}
K					X_{665}	X_{187}	
G_e						X_{650}	X_{3002}

Remarks. (1) X_{3002} is the intersection of the Brocard axis and the trilinear polar of the Gergonne point. It has coordinates

$$(a^2(a^3(b^2+c^2)-a^2(b+c)(b-c)^2-a(b^4+c^4)+(b+c)(b-c)^2(b^2+c^2)) : \cdots : \cdots).$$

(2) X_{3003} is the intersection of the orthic and Brocard axes. It has coordinates

$$(a^2(a^4(b^2+c^2)-2a^2(b^4-b^2c^2+c^4)+(b^2-c^2)^2(b^2+c^2)) : \cdots : \cdots).$$

The center of the rectangular hyperbola through E is X_{113}, the inferior of X_{74} on the the circumcircle.

Here are some examples of cevian products with very simple coordinates. They do not appear in the current edition of [7].

P_0	P_1	first barycentric coordinate of $T(P_0,\ P_1)$
G	X_9	$(a(b-c)(b+c-a)^2$
G	X_{56}	$(a^2(b-c)(a(b+c)+b^2+c^2)$
O	X_{21}	$(a^3(b-c)(b+c-a)(b^2+c^2-a^2)^2$
O	X_{55}	$(a^3(b-c)(b+c-a)^2(b^2+c^2-a^2)$
O	X_{56}	$(a^3(b-c)(b^2+c^2-a^2)$
K	N_a	$(a(b-c)(b+c-a)(a(b+c)+b^2+c^2)$
K	X_{99}	$(a^2(a^2(b^2+c^2)-2b^2c^2)$
G_e	X_{56}	$\frac{a^2(b^2-c^2)}{b+c-a}$
G_e	X_{57}	$\frac{a(b-c)}{b+c-a}$
N_a	X_{55}	$(a^2(b^2-c^2)(b+c-a)$
X_{21}	X_{55}	$(a^3(b-c)(b+c-a)$
X_{56}	X_{57}	$\frac{a^2(b-c)}{b+c-a}$

Remark. $T(X_{21},\ X_{55}) = T(X_{21},\ X_{56}) = T(X_{55},\ X_{56})$.

3. Inscribed triangles which circumscribe a given anticevian triangle

Proposition 5. *Let P be a given point with anticevian triangle $X'Y'Z'$. If XYZ is an inscribed triangle of ABC (with X, Y, Z on the sidelines BC, CA, AB respectively) such that X', Y', Z' lie on the lines YZ, ZX, XY respectively,* i.e.,

$X'Y'Z'$ is an inscribed triangle of XYZ, then XYZ is the cevian triangle of a point on the circumconic with perspector P.

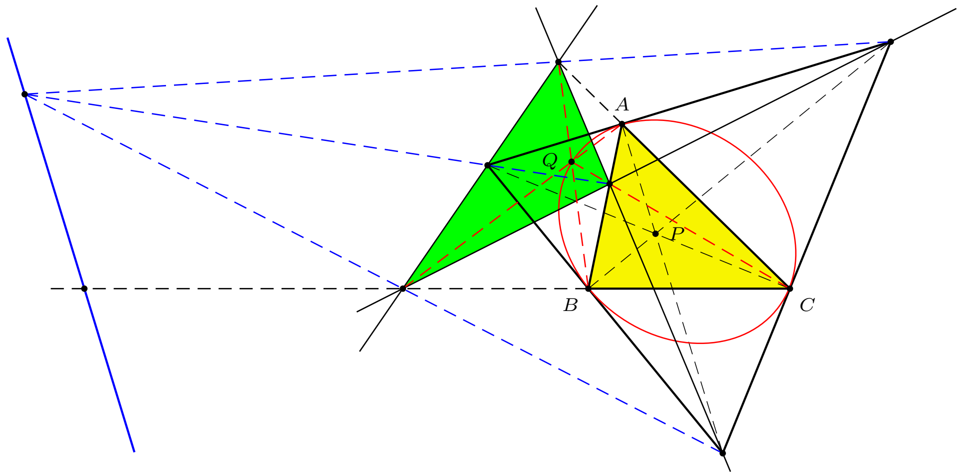

Figure 4.

Proof. Let $P = (u : v : w)$ so that

$$X' = (-u : v : w), \qquad Y' = (u : -v : w), \qquad Z' = (u : v : -w).$$

Since XYZ is an inscribed triangle of ABC,

$$X = (0 : t_1 : 1), \quad Y = (1 : 0 : t_2), \quad Z = (t_3 : 1 : 0),$$

for real numbers t_1, t_2, t_3. Here we assume that X, Y, Z do not coincide with the vertices of ABC. Since the lines YZ, ZX, XY contain the points respectively, we have

$$\begin{aligned} t_2u + t_2t_3v + w &= 0, \\ u + t_3v + t_3t_1w &= 0, \\ t_1t_2u + v + t_1w &= 0. \end{aligned}$$

From these,

$$0 = \begin{vmatrix} t_2 & t_2t_3 & 1 \\ 1 & t_3 & t_3t_1 \\ t_1t_2 & 1 & t_1 \end{vmatrix} = (t_1t_2t_3 - 1)^2.$$

It follows from the Ceva theorem that the lines AX, BY, CZ are concurrent. The inscribed triangle XYZ is the cevian triangle of a point $(p : q : r)$. The three collinearity conditions all reduce to

$$uqr + vrp + wpq = 0.$$

This means that $(p : q : r)$ lies on the circumconic with perspector $(u : v : w)$. $\square$

Proposition 6. *The locus of the perspector of the anticevian triangle of P and the cevian triangle of a point Q on the circumconic with perspector P is the trilinear polar of P.*

Proof. Let $Q = (u : v : w)$ be a point on the circumconic. The perspector is the cevian quotient

$$\left(p\left(-\frac{p}{u}+\frac{q}{v}+\frac{r}{w}\right) : \ q\left(\frac{p}{u}-\frac{q}{v}+\frac{r}{w}\right) : r\left(\frac{p}{u}+\frac{q}{v}-\frac{r}{w}\right)\right)$$
$$= (p(-pvw+qwu+ruv) : \ q(pvw-qwu+ruv) : \ r(pvw+qwu-ruv)).$$

Since $pvw + qwu + ruv = 0$, this simplifies into $(p^2vw : q^2wu : r^2uv)$, which clearly lies on the line $\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=0$, the trilinear polar of P. □

4. Brianchon-Poncelet theorem

For $P_0 = H$, the orthocenter, and $P_1 = X_7$, the Gergonne point, we have $T(P_0, P_1) = X_{650}$. The circumconic through P_0 and P_1 is

$$a(b-c)(b+c-a)yz + b(c-a)(c+a-b)zx + c(a-b)(a+b-c)xy = 0,$$

the Feuerbach conic, which is the isogonal conjugate of the line OI, and has center at the Feuerbach point

$$X_{11} = ((b-c)^2(b+c-a) : \ (c-a)^2(c+a-b) : \ (a-b)^2(a+b-c)).$$

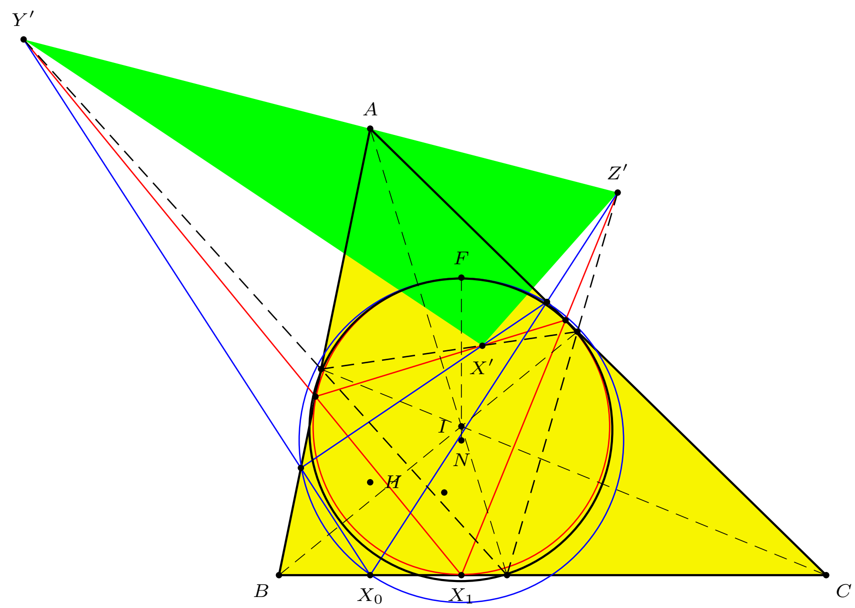

Figure 5.

The theorem of Emelyanov and Emelyanova therefore can be generalized as follows: *the the cevian circumcircle of a point on the Feuerbach hyperbola contains the Feuerbach point*. This in turn is a special case of a celebrated theorem of Brianchon and Poncelet in 1821.

Theorem 7 (Brianchon-Poncelet [1])**.** *Given a point P, the cevian circumcircle of an arbitrary point on the rectangular circum-hyperbola through P (and the orthocenter H) contains the center of the hyperbola which is on the nine-point circle of the reference triangle.*

At the end of their paper Brianchon and Poncelet made a remarkable conjecture about the locus of the centers of conics through four given points. This was subsequently proved by J. D. Gergonne [6].

Theorem 8 (Brianchon-Poncelet-Gergonne)**.** *The locus of the centers of conics through four given points in general positions in a plane is a conic through*
(i) the midpoints of the six segments joining them and
(ii) the intersections of the three pairs of lines joining them two by two.

Proposition 9. *The cevian circumcircle of a point on a nondegenerate circumconic contains the center of the conic if and only if the conic is a rectangular hyperbola.*

Proof. (a) The sufficiency part follows from Theorem 7.

(b) For the converse, consider a nondegenerate conic through A, B, C, P whose center W lies on the cevian circumcircle of P. The locus of centers of conics through A, B, C, P is, by Theorem 8, a conic $\mathcal{C}$ through the traces of P on the sidelines of triangle ABC. The four common points of $\mathcal{C}$ and the cevian circumcircle of P are the traces of P and W. By (a), the cevian circumcircle of P contains the center of the rectangular circum-hyperbola through P, which must coincide with W. Therefore the conic in question in rectangular. $\square$

Since the Feuerbach hyperbola contain the incenter I, we have the following result. See Figure 5.

Corollary 10. *The cevian circumcircle of the incenter contains the Feuerbach point.*

Applying Brianchon-Poncelet theorem to the Kiepert perspectors, we obtain the following interesting result.

Corollary 11. *Given triangle ABC, construct on the sides similar isosceles triangles BCX', CAY', and ABZ'. Let AX', BY', CZ' intersect BC, CA, AB at X, Y, Z respectively. The circle through X, Y, Z also contains the center X_{115} of the Kiepert hyperbola, which is also the midpoint between the two Fermat points.*

5. Second tangents to an inscribed conic from the traces of a point

Consider an inscribed conic $\mathcal{C}_0$ with Brianchon point $P_0 = (u_0 : v_0 : w_0)$, so that its equation is

$$\left(\frac{x}{u_0}\right)^2 + \left(\frac{y}{v_0}\right)^2 + \left(\frac{z}{w_0}\right)^2 - \frac{2yz}{v_0w_0} - \frac{2zx}{w_0u_0} - \frac{2xy}{u_0v_0} = 0.$$

Let $P_1 = (u_1 : v_1 : w_1)$ be a given point with cevian triangle $X_1Y_1Z_1$. The sidelines of triangle ABC are tangents from X_1, Y_1, Z_1 to the conic $\mathcal{C}_0$. From each of these points there is a second tangent to the conic. J.-P. Ehrmann [5] has computed the second points of tangency X_2, Y_2, Z_2, and concluded that the triangle $X_2Y_2Z_2$ is perspective with ABC at the point

$$\left(\frac{u_1^2}{u_0} : \frac{v_1^2}{v_0} : \frac{w_1^2}{w_0}\right).$$

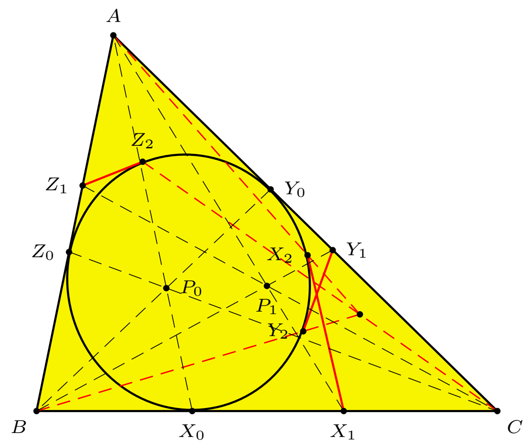

Figure 6.

More precisely, the coordinates of X_2, Y_2, Z_2 are as follows.

$$X_2 = \left(u_0\left(\frac{v_1}{v_0} - \frac{w_1}{w_0}\right)^2 : \frac{v_1^2}{v_0} : \frac{{w_1}^2}{w_0}\right),$$

$$Y_2 = \left(\frac{u_1^2}{u_0} : v_0\left(\frac{w_1}{w_0} - \frac{v_1}{v_0}\right)^2 : \frac{{w_1}^2}{w_0}\right),$$

$$Z_2 = \left(\frac{u_1^2}{u_0} : \frac{v_1^2}{v_0} : w_0\left(\frac{u_1}{u_0} - \frac{v_1}{v_0}\right)^2\right).$$

Proposition 12. *The lines Y_0Z_0, Y_1Z_1, Y_2Z_2 are concurrent; similarly for the triples Z_0X_0, Z_1X_1, Z_2X_2 and X_0Y_0, X_1Y_1, X_2Y_2.*

Proof. The line Y_2Z_2 has equation

$$u_1\left(\frac{x}{u_0}\left(-\frac{u_1}{u_0} + \frac{v_1}{v_0} + \frac{w_1}{w_0}\right) + \frac{y}{v_0}\left(\frac{u_1}{u_0} - \frac{v_1}{v_0} + \frac{w_1}{w_0}\right) + \frac{z}{w_0}\left(\frac{u_1}{u_0} + \frac{v_1}{v_0} - \frac{w_1}{w_0}\right)\right)$$
$$- \frac{2v_1w_1x}{v_0w_0} = 0.$$

With

$$(x : y : z) = \left(-u_1\left(\frac{v_1}{v_0} - \frac{w_1}{w_0}\right) : v_1\left(\frac{w_1}{w_0} - \frac{u_1}{u_0}\right) : w_1\left(\frac{u_1}{u_0} - \frac{v_1}{v_0}\right)\right),$$

we have, apart from a factor u_1,

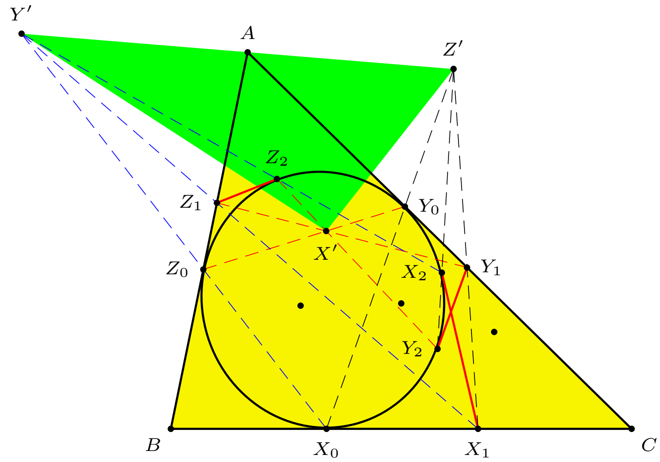

Figure 7.

$$
\begin{aligned}
&-\frac{u_1}{u_0}\left(\frac{v_1}{v_0}-\frac{w_1}{w_0}\right)\left(-\frac{u_1}{u_0}+\frac{v_1}{v_0}+\frac{w_1}{w_0}\right)+\frac{v_1}{v_0}\left(\frac{w_1}{w_0}-\frac{u_1}{u_0}\right)\left(\frac{u_1}{u_0}-\frac{v_1}{v_0}+\frac{w_1}{w_0}\right)\\
&+\frac{w_1}{w_0}\left(\frac{u_1}{u_0}-\frac{v_1}{v_0}\right)\left(\frac{u_1}{u_0}+\frac{v_1}{v_0}-\frac{w_1}{w_0}\right)+\frac{2v_1w_1}{v_0w_0}\left(\frac{v_1}{v_0}-\frac{w_1}{w_0}\right)\\
=&-\frac{2v_1w_1}{v_0w_0}\left(\frac{v_1}{v_0}-\frac{w_1}{w_0}\right)+\frac{2v_1w_1}{v_0w_0}\left(\frac{v_1}{v_0}-\frac{w_1}{w_0}\right)\\
=&\,0.
\end{aligned}
$$

This shows that the line Y_2Z_2 contains the point $X' = Y_0Z_0 \cap Y_1Z_1$. □

We conclude with some examples of the triangle centers from the inscribed conics with given perspectors P_0 and P_1. In the table below,

$$Q_{0,1} = \left(\frac{u_0^2}{u_1} : \frac{v_0^2}{v_1} : \frac{w_0^2}{w_1}\right) \quad \text{and} \quad Q_{1,0} = \left(\frac{u_1^2}{u_0} : \frac{v_1^2}{v_0} : \frac{w_1^2}{w_0}\right).$$

P_0	P_1	$T(P_0, P_1)$	$P_0/(T(P_0, P_1))$	$P_1/(T(P_0, P_1))$	$Q_{0,1}$	$Q_{1,0}$
G	H	X_{523}	X_{125}	X_{115}	X_{69}	X_{393}
G	K	X_{512}	Q_1	X_{1084}	X_{76}	X_{32}
G	G_e	X_{514}	X_{11}	X_{1086}	X_8	X_{279}
G	N_a	X_{522}	X_{11}	X_{1146}	X_7	X_{346}
H	K	X_{647}	Q_2	Q_3	X_{2052}	X_{184}
H	G_e	X_{650}	X_{3022}	Q_4	X_{1857}	Q_5
H	N_a	X_{650}	Q_6	Q_4	X_{1118}	X_{1265}
K	G_e	X_{665}	Q_7	Q_8	X_{2175}	Q_9
G_e	N_a	X_{650}	Q_6	X_{3022}	X_{479}	Q_{10}

The new triangle centers Q_i have simple coordinates given below.

Q_1	$a^2(b^2-c^2)^2$
Q_2	$a^2(b^2-c^2)^2(b^2+c^2-a^2)^2$
Q_3	$a^4(b^2-c^2)^2(b^2+c^2-a^2)^3$
Q_4	$a^2(b-c)^2(b+c-a)^2(b^2+c^2-a^2)$
Q_5	$\frac{b^2+c^2-a^2}{(b+c-a)^2}$
Q_6	$a^2(b-c)^2(b+c-a)$
Q_7	$a^4(b-c)^2(b+c-a)(a(b+c)-(b^2+c^2))^2$
Q_8	$a^2(b-c)^2(a(b+c)-(b^2+c^2))^2$
Q_9	$\frac{1}{a^2(b+c-a)^2}$
Q_{10}	$(b+c-a)^3$

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