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# Constructive Solution of a Generalization of Steinhaus' Problem on Partition of a Triangle

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**Abstract**. We present a constructive solution to a generalization of Hugo Steinhaus' problem of partitioning a given triangle, by dropping perpendiculars from an interior point, into three quadrilaterals whose areas are in prescribed proportions.

## 1. Generalized Steinhaus problem

Given an acute angled triangle $ABC$, Steinhaus' problem asks a point $P$ in its interior with pedals $P_a$, $P_b$, $P_c$ on $BC$, $CA$, $AB$ such that the quadrilaterals $AP_bPP_c$, $BP_cP_a$, and $CP_aPP_b$ have equal areas. See [3] and the bibliographic information therein. A. Tyszka [2] has also shown that Steinhaus' problem is in general not soluble by ruler-and-compass. We present a simple constructive solution (using conics) of a generalization of Steinhaus' problem. In this note, the area of a polygon $\mathscr{P}$ will be denoted by $\Delta(\mathscr{P})$. In particular, $\Delta = \Delta(ABC)$. Thus, given three positive real numbers $u$, $v$, $w$, we look for the point(s) $P$ such that
(1) $P$ is inside $ABC$ and $P_a$, $P_b$, $P_c$ lie respectively in the segments $BC$, $CA$, $AB$,
(2) $\Delta(AP_bPP_c) : \Delta(BP_cPP_a) : \Delta(CP_aPP_b) = u : v : w$.

We do not require the triangle to be acute-angled.

**Lemma 1.** *Consider a point $P$ inside the angular sector bounded by the half-lines $AB$ and $AC$, with projections $P_b$ and $P_c$ on $AC$ and $AB$ respectively. For a positive real number $k$, $\Delta(AP_bPP_c) = k \cdot \Delta(ABC)$ if and only if $P$ lies on the rectangular hyperbola with center $A$, focal axis the internal bisector $AI$, and semi-major axis $\sqrt{kbc}$.*

*Proof.* We take $A$ for pole and the bisector $AI$ for polar axis; let $(\rho,\ \theta)$ be the polar coordinates of $P$. As $AP_b = \rho\cos\left(\frac{A}{2} - \theta\right)$ and $PP_b = \rho\sin\left(\frac{A}{2} - \theta\right)$, we have $\Delta(APP_b) = \frac{1}{2}\rho^2\sin(A - 2\theta)$. Similarly, $\Delta(AP_cP) = \frac{1}{2}\rho^2\sin(A + 2\theta)$. Hence the quadrilateral $AP_bPP_c$ has area $\frac{1}{2}\rho^2\sin A\cos 2\theta$. Therefore,

$$\Delta(AP_bPP_c) = k \cdot \Delta(ABC) \Longleftrightarrow \rho^2\cos 2\theta = \frac{2k \cdot \Delta(ABC)}{\sin A} = kbc.$$

□

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**Theorem 2.** *Let $U$ be the point with barycentric coordinates $(u : v : w)$ and $M_1$, $M_2$, $M_3$ be the antipodes on the circumcircle $\Gamma$ of $ABC$ of the points whose Simson lines pass through $U$ and $P$ the incenter of the triangle $M_1M_2M_3$. If $P$ verifies* (1)*, then $P$ is the unique solution of our problem. Otherwise, the generalized Steinhaus problem has no solution.*

*Remarks.* (a) Of course, if $ABC$ is acute angled, and $P$ inside $ABC$, then (1) will be verified.

(b) As $U$ lies inside the Steiner deltoid, there exist three real Simson lines through $U$; so $M_1$, $M_2$, $M_3$ are real and distinct.

(c) Let $h_A$ be the rectangular hyperbola with center $A$, focal axis $AI$, and semi-major axis $\sqrt{\dfrac{u}{u+v+w}} \cdot bc$, and define rectangular hyperbolas $h_B$ and $h_C$ analogously.

If $P$ verifies (1), it will verify (2) if and only if $P \in h_A \cap h_B$. In this case, $P \in h_C$, and the solutions of our problem are the common points of $h_A$, $h_B$, $h_C$ verifying (1).

(d) The four common points $P_1$, $P_2$, $P_3$, $P_4$ (real or imaginary) of the rectangular hyperbolae $h_A$, $h_B$, $h_C$ form an orthocentric system. As $h_A$, $h_B$, $h_C$ are centered respectively at $A$, $B$, $C$, any conic through $P_1$, $P_2$, $P_3$, $P_4$ is a rectangular hyperbola with center on $\Gamma$. As the vertices of the diagonal triangle of this orthocentric system are the centers of the degenerate conics through $P_1$, $P_2$, $P_3$, $P_4$ , they lie on $\Gamma$.

(e) We will see later that $P_1$, $P_2$, $P_3$, $P_4$ are always real.

## 2. Proof of Theorem 2

If $P$ has homogeneous barycentric coordinates $(x : y : z)$ with reference to triangle $ABC$, then

$$(x+y+z)^2\Delta(APP_b) = y\left(z + \frac{b^2+c^2-a^2}{2b^2}y\right)\Delta,$$
$$(x+y+z)^2\Delta(AP_cP) = z\left(y + \frac{b^2+c^2-a^2}{2c^2}z\right)\Delta,$$

where $\Delta = \Delta(ABC)$. Hence the barycentric equation of $h_A$ is

$$h_A(x,y,z) := \frac{b^2+c^2-a^2}{2}\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) + 2\,y\,z - \frac{u}{u+v+w}\,(x+y+z)^2 = 0.$$

We get $h_B$ and $h_C$ by cyclically permuting $a$, $b$, $c$; $u$, $v$, $w$; $x$, $y$, $z$.

If $M = (x : y : z)$ is a vertex of the diagonal triangle of $P_1P_2P_3P_4$, it has the same polar line (the opposite side) with respect to the three conics $h_A$, $h_B$, $h_C$. Hence,

$$\frac{\partial h_B}{\partial y}\frac{\partial h_C}{\partial z} - \frac{\partial h_B}{\partial z}\frac{\partial h_C}{\partial y} = \frac{\partial h_C}{\partial z}\frac{\partial h_A}{\partial x} - \frac{\partial h_C}{\partial x}\frac{\partial h_A}{\partial z} = \frac{\partial h_A}{\partial x}\frac{\partial h_B}{\partial y} - \frac{\partial h_A}{\partial y}\frac{\partial h_B}{\partial x} = 0.$$

Let $N$ be the reflection of $M$ in the circumcenter $O$; $N_aN_bN_c$ the pedal triangle of $N$. Clearly, $N_a$, $N_b$, $N_c$ are the reflections of the vertices of the pedal triangle of $M$ in the midpoints of the corresponding sides of $ABC$. Now, $N_b$ and $N_c$ have coordinates

$$(b^2+c^2-a^2)y+2b^2z:\ 0:\ (a^2+b^2-c^2)y+2b^2x$$

and

$$(b^2+c^2-a^2)z+2c^2y:\ (c^2+a^2-b^2)z+2c^2x:\ 0$$

respectively. A straightforward computation shows that

$$\det[N_b,\ N_c,\ U]=b^2c^2\,(u+v+w)\left(\frac{\partial h_B}{\partial y}\frac{\partial h_C}{\partial z}-\frac{\partial h_B}{\partial z}\frac{\partial h_C}{\partial y}\right)=0.$$

Similarly, $\det[N_c,\ N_a,\ U]=\det[N_a,\ N_b,\ U]=0$. It follows that $N$ lies on the circumcircle (we knew that already by Remark (d)), and the Simson line of $N$ passes through $U$.

Hence, $M_1M_2M_3$ is the diagonal triangle of the orthocentric system $P_1P_2P_3P_4$, which means that $P_1P_2P_3P_4$ are real and are the incenter and the three excenters of $M_1M_2M_3$.

As the three excenters of a triangle lie outside his circumcircle, the incenter of $M_1M_2M_3$ is the only common point of $h_A$, $h_B$, $h_C$ inside $\Gamma$. This completes the proof of Theorem 2.

## 3. Constructions

In [1], the author has given a construction of the points on the circumcircle whose Simson line pass through a given point. Let $U^-$ and $U^+$ be the complement and the anticomplement of $U$, *i.e.*, the images of $U$ under the homotheties $\mathsf{h}\left(G,-\frac{1}{2}\right)$ and $\mathsf{h}(G,-2)$ respectively. Since

$$(\text{Reflection in } O)\ \circ(\text{Translation by } \overrightarrow{HU})=\text{Reflection in } U^-,$$

if $h_0$ is the reflection in $U^-$ of the rectangular circumhyperbola through $U$, and $M_4$ the antipode of $U^+$ on $h_0$, then $M_1$, $M_2$, $M_3$, $M_4$ are the four common points of $h_0$ and the circumcircle.

In the case $u=v=w=1$, $h_0$ is the reflection in the centroid $G$ of the Kiepert hyperbola of $ABC$. It intersects the circumcircle $\Gamma$ at $M_1$, $M_2$, $M_3$ and the Steiner point of $ABC$. See Figure 1.

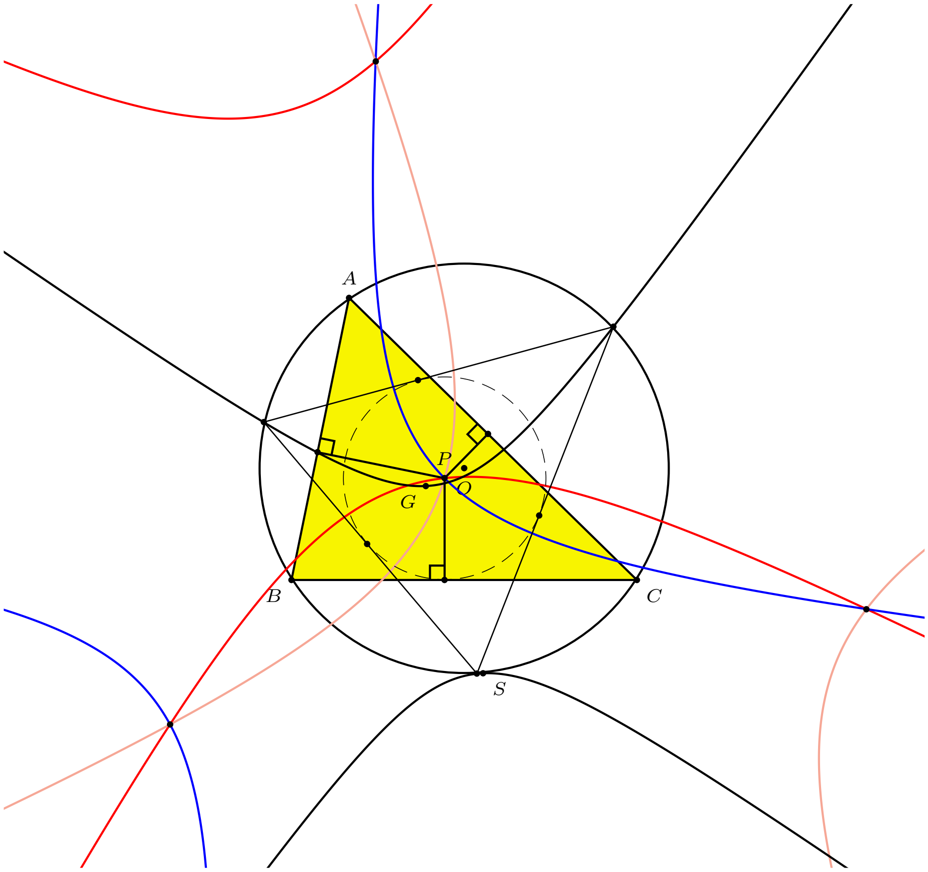


Figure 1.

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