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Cyclic Quadrilaterals with Prescribed Varignon Parallelogram

Michel Bataille

Abstract. We prove that the vertices of a given parallelogram $\mathcal{P}$ are the midpoints of the sides of infinitely many *cyclic* quadrilaterals and show how to construct such quadrilaterals. Then we discuss some of their properties and identify related loci. Lastly, the cases when $\mathcal{P}$ is a rectangle or a rhombus are examined.

1. Introduction

The following well-known theorem of elementary geometry, attributed to the French mathematician Pierre Varignon (1654-1722), was published in 1731: if A, B, C, D are four points in the plane, the respective midpoints P, Q, R, S of AB, BC, CD, DA are the vertices of a parallelogram. We will say that $PQRS$ is the Varignon parallelogram of $ABCD$, in short $PQRS = \mathcal{V}(ABCD)$. In a converse way, given a parallelogram $\mathcal{P}$, there exist infinitely many quadrilaterals $ABCD$ such that $\mathcal{P} = \mathcal{V}(ABCD)$. In §2, we offer a quick review of this general result, introducing the diagonal midpoints of $ABCD$ which are of constant use afterwards. The primary result of this paper, namely that infinitely many of these quadrilaterals $ABCD$ are cyclic, is proved in §3 and the proof leads naturally to a construction of such quadrilaterals. Further results, including a simpler construction, are established in §4, all centering on a rectangular hyperbola determined by $\mathcal{P}$. Finally, §5 is devoted to particular results that hold if $\mathcal{P}$ is either a rectangle or a rhombus.

In what follows, $\mathcal{P} = PQRS$ denotes a parallelogram whose vertices are not collinear. The whole work takes place in the plane of $\mathcal{P}$.

2. Quadrilaterals $ABCD$ with $\mathcal{P} = \mathcal{V}(ABCD)$

The construction of a quadrilateral $ABCD$ satisfying $\mathcal{P} = \mathcal{V}(ABCD)$ is usually presented as follows: start with an arbitrary point A and construct successively the symmetric B of A about P, the symmetric C of B about Q and the symmetric D of C about R (see Figure 1). Because $\mathcal{P}$ is a parallelogram, A is automatically the symmetric of D about S and $ABCD$ is a solution (see [1, 2]).

Let M, M' be the midpoints of the diagonals of $ABCD$ (in brief, the diagonal midpoints of $ABCD$) and let O be the center of $\mathcal{P}$. Since $4O = 2P + 2R = A + B + C + D = 2M + 2M'$, the midpoint of MM' is O. This simple property allows another construction of $ABCD$ from $\mathcal{P}$ that will be preferred in the next sections: start with two points M, M' symmetric about O; then obtain A, C such that $\overrightarrow{AM} = \overrightarrow{PQ} = \overrightarrow{MC}$ and B, D such that $\overrightarrow{BM'} = \overrightarrow{QR} = \overrightarrow{M'D}$. Exchanging the roles of M, M' provides another solution $A'B'C'D'$ with the same set $\{M, M'\}$ of diagonal

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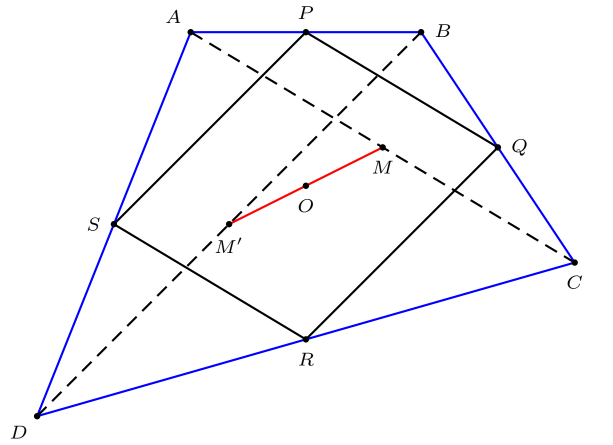

Figure 1

midpoints (see Figure 2). Clearly, $ABCD$ and $A'B'C'D'$ are symmetrical about O.

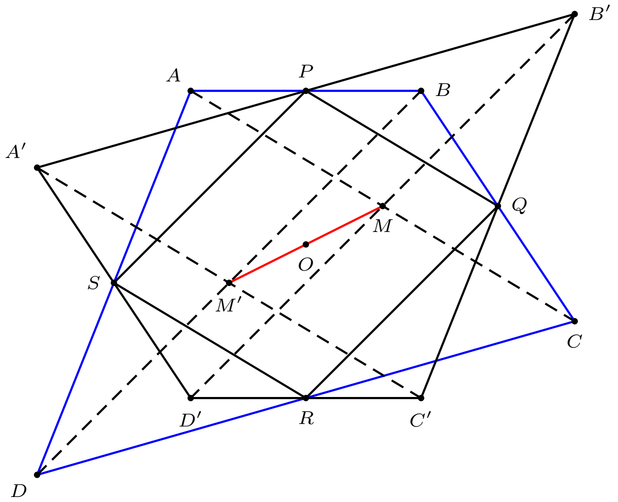

Figure 2

3. Cyclic quadrilaterals $ABCD$ with $\mathcal{P} = \mathcal{V}(ABCD)$

The previous section has brought out the role of diagonal midpoints when looking for quadrilaterals $ABCD$ such that $\mathcal{P} = \mathcal{V}(ABCD)$. We characterize the diagonal midpoints of *cyclic* solutions and show how to construct them from $\mathcal{P}$, obtaining the following theorem.

Theorem 1. *Given $\mathcal{P}$, there exist infinitely many cyclic quadrilaterals $ABCD$ such that $\mathcal{P} = \mathcal{V}(ABCD)$. Such quadrilaterals can be constructed from $\mathcal{P}$ by ruler and compass.*

Proof. Consider a Cartesian system with origin at O and x-axis parallel to PQ (see Figure 3). The affix of a point Z is denoted by z. For example, $q-p$ is a real number.

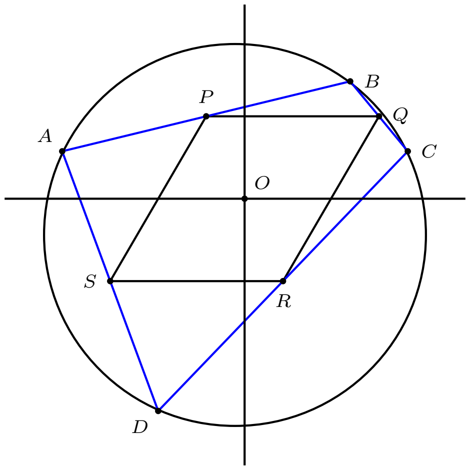

Figure 3

Let $ABCD$ be such that $\mathcal{P} = \mathcal{V}(ABCD)$. Then $A \neq C$, $B \neq D$ and the quadrilateral $ABCD$ is cyclic if and only if the cross-ratio $\rho = \frac{d-a}{d-b} \cdot \frac{c-b}{c-a}$ is a real number. With $b = 2p - a$, $c = 2q - 2p + a$, $d = -2q - a$ and allowing for $q - p \in \mathbb{R}$, the calculation of ρ yields the condition:

$$(q-p+a)^2 = p^2 + \lambda(p+q)$$

for some real number λ. Thus, $ABCD$ is cyclic if and only if the affixes m, $m' = -m$ of its diagonal midpoints M, M' are the square roots of a complex number of the form $p^2 + \lambda(p+q)$, where $\lambda \in \mathbb{R}$. Clearly, distinct values λ_1, λ_2 for λ lead to corresponding disjoint sets $\{M_1, M_1'\}, \{M_2, M_2'\}$ of diagonal midpoints, hence to distinct solutions for cyclic quadrilaterals. It follows that our problem has infinitely many solutions.

Consider P_2 with affix p^2 and choose a point K on the line through P_2 parallel to QR. The affix k of K is of the form $p^2 + \lambda(p+q)$ with $\lambda \in \mathbb{R}$. The construction of the corresponding pair M, M' is straightforward and achieved in Figure 4 where for the sake of simplification we take OP as the unit of length: M, M' are on the angle bisector of $\angle xOK$ and $OM = OM' = \sqrt{OK}$ (we skip the classical construction of the square root of a given length). $\square$

Exchanging the roles of M and M' (as in §2) evidently gives a solution inscribed in the symmetric of the circle $(ABCD)$ about O. In §4, we will indicate a different construction of suitable diagonal midpoints M, M'.

4. The rectangular hyperbola $\mathcal{H}(\mathcal{P})$

With the aim of obtaining the diagonal midpoints M, M' more directly, it seems interesting to identify their locus as the real number λ varies. This brings to light

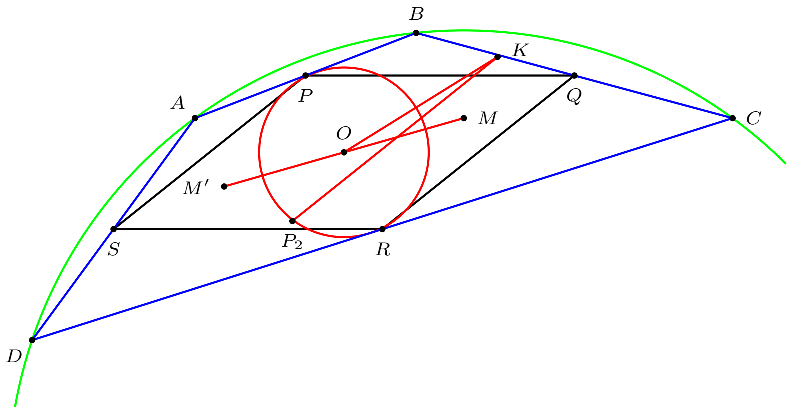

Figure 4

an unexpected hyperbola which will also provide more results about our quadrilaterals.

Theorem 2. *Consider the cyclic quadrilaterals $ABCD$ such that $\mathcal{P} = \mathcal{V}(ABCD)$. If $\mathcal{P}$ is not a rhombus, the locus of their diagonal midpoints is the rectangular hyperbola $\mathcal{H}(\mathcal{P})$ with the same center O as $\mathcal{P}$, passing through the vertices P, Q, R, S of $\mathcal{P}$. If $\mathcal{P}$ is a rhombus, the locus is the pair of diagonals of $\mathcal{P}$.*

Proof. We use the same system of axes as in the preceding section and continue to suppose that $OP = 1$. We denote by θ the directed angle $\angle(\overrightarrow{SR}, \overrightarrow{SP})$ that is, $\theta = \arg(p+q)$. Note that $\sin\theta \neq 0$. Let $m = x + iy$ with $x, y \in \mathbb{R}$. From $m^2 = p^2 + \lambda(p+q)$, we obtain $(x+iy)^2 = e^{2it} + \lambda\mu e^{i\theta}$ where $t = \arg(p)$ and $\mu = |p+q|$ and we readily deduce:

$$x^2 - y^2 = \cos 2t + \lambda\mu\cos\theta, \qquad 2xy = \sin 2t + \lambda\mu\sin\theta.$$

The elimination of λ shows that the locus of M (and of M' as well) is the curve $\mathcal{C}$ with equation

$$x^2 - y^2 - 2(\cot\theta)xy + \nu = 0, \tag{1}$$

where $\nu = \cot\theta\sin 2t - \cos 2t = \frac{\sin(2t-\theta)}{\sin\theta}$. Thus, when $\nu \neq 0$, $\mathcal{C}$ is a rectangular hyperbola centered at O with asymptotes

(ℓ) $$y = x\tan(\theta/2),$$

and

(ℓ') $$y = -x\cot(\theta/2),$$

and $\mathcal{C}$ degenerates into these two lines if $\nu = 0$ (we shall soon see that the latter occurs if and only if $\mathcal{P}$ is a rhombus). Note that (ℓ) and (ℓ') are the axes of symmetry of the medians of $\mathcal{P}$. An easy calculation shows that the coordinates $x_P = \cos t$, $y_P = \sin t$ of P satisfy (1), meaning that $P \in \mathcal{C}$. As for Q, the coordinates are $x_Q = \mu\cos\theta - \cos t$, $y_Q = \mu\sin\theta - \sin t$, but observing that

$y_Q = y_P$, we find $x_Q = 2\sin t \cot\theta - \cos t,\ y_Q = \sin t$. Again, x_Q, y_Q satisfy (1) and Q is a point of $\mathcal{C}$ as well. Thus, the parallelogram $\mathcal{P}$ is inscribed in $\mathcal{C}$. It follows that $\nu = 0$ if and only if (ℓ) and (ℓ') are the diagonals of $\mathcal{P}$. Since (ℓ) and (ℓ') are perpendicular, the situation occurs if $\mathcal{P}$ is a rhombus and only in that case. Otherwise, $\mathcal{C}$ is the rectangular hyperbola $\mathcal{H}(\mathcal{P})$, as defined in the statement of the theorem (see Figure 5). $\square$

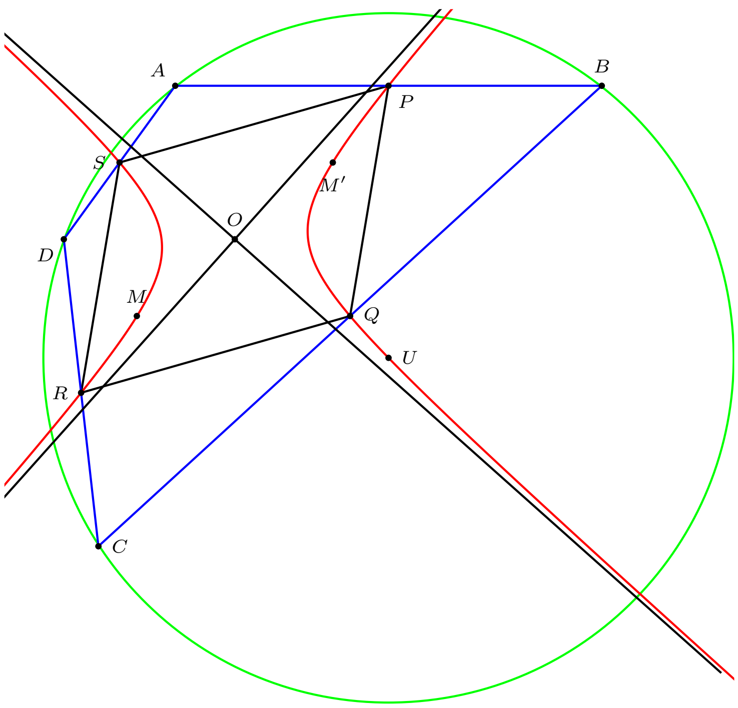

Figure 5

Figure 5 shows the center U of the circle through A, B, C, D as a point of $\mathcal{H}(\mathcal{P})$. This is no coincidence! Being the circumcenter of ΔABC, U is also the orthocenter of its median triangle MPQ. Since the latter is inscribed in $\mathcal{H}(\mathcal{P})$, a well-known property of the rectangular hyperbola ensures that its orthocenter is on $\mathcal{H}(\mathcal{P})$ as well. Conversely, any point U of $\mathcal{H}(\mathcal{P})$ can be obtained in this way by taking for M the orthocenter of ΔUPQ. We have proved:

Theorem 3. *If $\mathcal{P}$ is not a rhombus, $\mathcal{H}(\mathcal{P})$ is the locus of the circumcenter of a cyclic quadrilateral $ABCD$ such that $\mathcal{P} = \mathcal{V}(ABCD)$.*

Of course, if $\mathcal{P}$ is a rhombus, the locus is the pair of diagonals of $\mathcal{P}$.

As another consequence of Theorem 2, we give a construction of a pair M, M' of diagonal midpoints simpler than the one in §3: through a vertex of $\mathcal{P}$, say Q, draw a line intersecting (ℓ) and (ℓ') at W and W'. As is well-known, the symmetric M of Q about the midpoint of WW' is on $\mathcal{H}(\mathcal{P})$. This point M and its symmetric M' about O provide a suitable pair. In addition, the orthocenter of ΔMPQ is the center U of the circumcircle of $ABCD$ (see Figure 6).

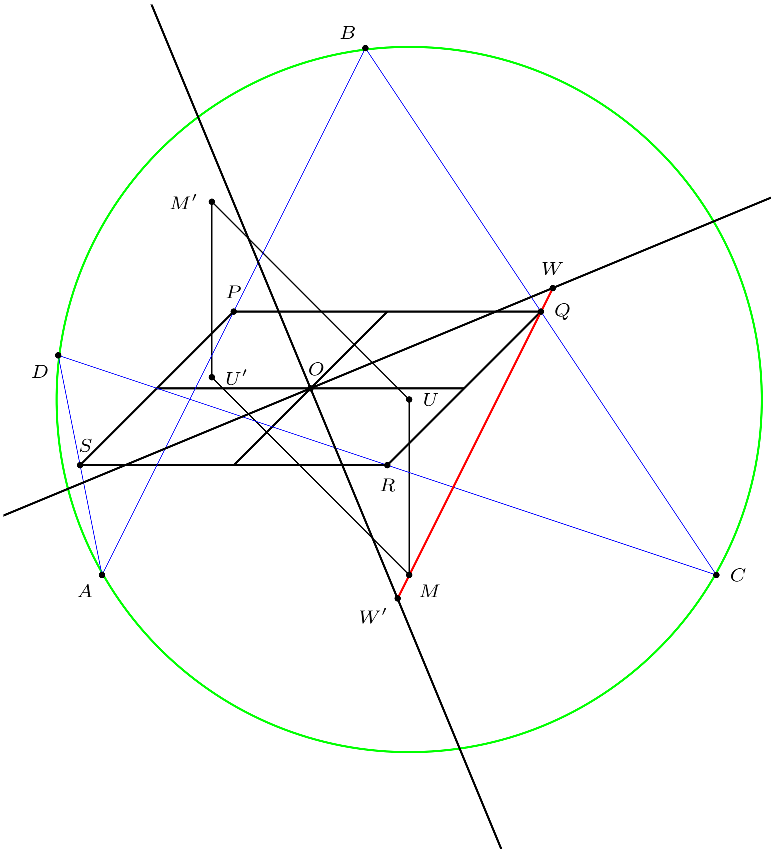

Figure 6

We shall end this section with a remark about the circumcenter U' of the quadrilateral $A'B'C'D'$ which shares the diagonal midpoints M, M' of $ABCD$ (as seen in §2). Clearly, $UMU'M'$ is a parallelogram with center O, inscribed in $\mathcal{H}(\mathcal{P})$ (Figure 6). Since UM and UM' are respectively perpendicular to PQ and PS, the directed angles of lines $\angle(UM, UM')$ and $\angle(PQ, PS)$ are equal (modulo π). Thus, $UMU'M'$ and $\mathcal{P}$ are equiangular.

5. Special cases

First, suppose that $\mathcal{P}$ is a rectangle and consider a cyclic quadrilateral $ABCD$ such that $\mathcal{P} = \mathcal{V}(ABCD)$. From the final remark of the previous section, $UMU'M'$ is a rectangle and since UM is perpendicular to PQ, the sides of $UMU'M'$ are parallel to those of $\mathcal{P}$. Recalling that M is on AC and M' on BD, we conclude that

U' is the point of intersection of the (perpendicular) diagonals of $ABCD$. Now, suppose that AC intersects PS at A_1, QR at C_1 and that BD intersects PQ at B_1, RS at D_1 (see Figure 7). Obviously, A_1, B_1, C_1 and D_1 are the midpoints of $U'A$, $U'B$, $U'C$ and $U'D$, so that A_1, B_1, C_1, D_1 are on the circle image of $(ABCD)$ under the homothety with center U' and ratio $\frac{1}{2}$. Since $\overrightarrow{U'O} = \frac{1}{2}\overrightarrow{U'U}$, the center of this circle $(A_1B_1C_1D_1)$ is just the center O of $\mathcal{P}$.

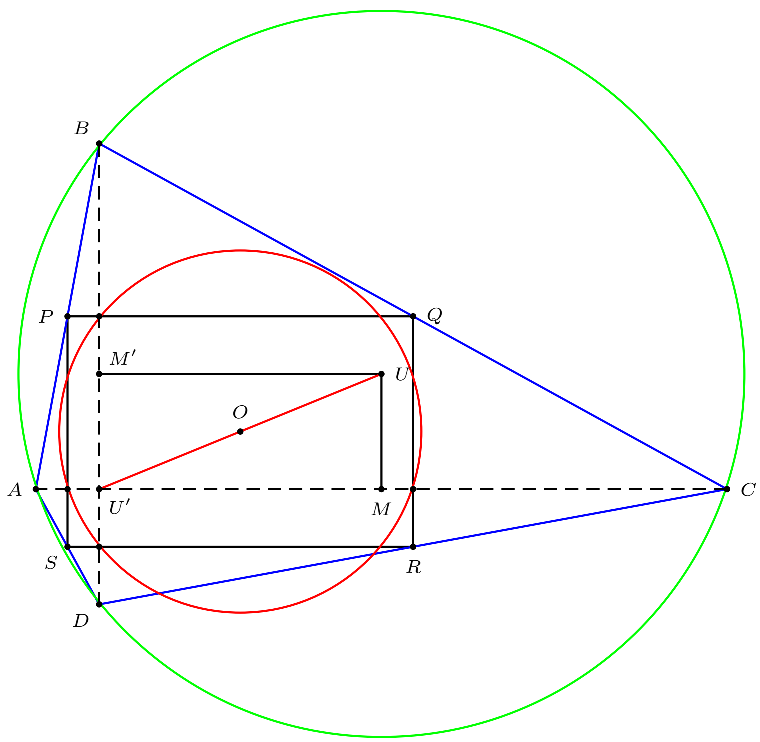

Figure 7

Conversely, draw any circle with center O intersecting the lines SP at A_1, A_1', PQ at B_1, B_1', QR at C_1, C_1' and RS at D_1, D_1', the notations being chosen so that A_1C_1, $A_1'C_1'$ are parallel to PQ and B_1D_1, $B_1'D_1'$ are parallel to QR. If $U' = A_1C_1 \cap B_1D_1$, then the image $ABCD$ of $A_1B_1C_1D_1$ under the homothety with center U' and ratio 2 is cyclic and satisfies $\mathcal{P} = \mathcal{V}(ABCD)$. For instance, because $U'A_1PB_1$ is a rectangle, P is the image of the midpoint of A_1B_1 and as such, is the midpoint of AB. The companion solution $A'B'C'D'$ is similarly obtained from $A_1'B_1'C_1'D_1'$.

Thus, in the case when $\mathcal{P}$ is a rectangle, a very quick construction provides suitable quadrilaterals $ABCD$. As a corollary of the analysis above, we have the following property that can also be proved directly:

Theorem 4. *If A, B, C, D are on a circle with center U and AC is perpendicular to BD at U', then the midpoint of UU' is the center of the rectangle $\mathcal{V}(ABCD)$.*

We conclude with a brief comment on the case when $\mathcal{P}$ is a rhombus. Remarking that if $\mathcal{P} = \mathcal{V}(ABCD)$, then $AC = 2PQ = 2QR = BD$, we see that any cyclic solution for $ABCD$ must be an isosceles trapezoid (possibly a self-crossing one). Conversely, if $ABCD$ is an isosceles trapezoid, then it is cyclic and $\mathcal{V}(ABCD)$ is a rhombus. The construction of a solution $ABCD$ from $\mathcal{P}$ simply follows from the choice of two points M, M' as diagonal midpoints of $ABCD$ on either diagonal of $\mathcal{P}$ (see Figure 8).

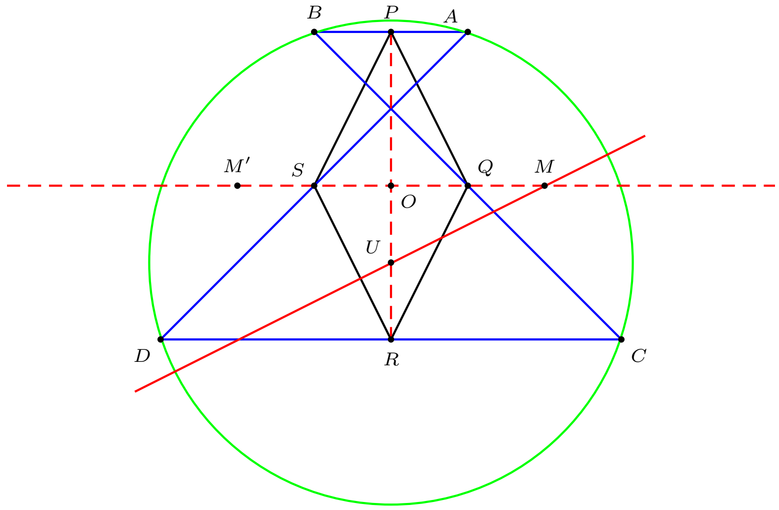

Figure 8

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Michel Bataille: 12 rue Sainte-Catherine, 76000 Rouen, France
E-mail address: michelbataille@wanadoo.fr