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Kronecker's Approximation Theorem and a Sequence of Triangles

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Abstract. We investigate the dynamic behavior of the sequence of nested triangles with a fixed division ratio on their sides. We prove a result concerning a special case that was not examined in [1]. We also provide an answer to an open problem posed in [3].

1. Introduction

The dynamic behavior of a sequence of polygons is an intriguing research area and many articles have been devoted to it (see e.g. [1], [2], [3] and the references therein). The questions that arise about these sequences are mainly two. The first one is about the existence of a limiting point of the sequence. The second one is about the dynamic behavior of the shapes of the polygons that belong to the sequence. Thus, it is possible to find a limiting shape, periodical shapes or an even more complicated behavior. In this article we are interesting for the sequence of triangles with a fixed division ratio on their sides. Let $A_0B_0C_0$ be an initial triangle and let the points A_1 on B_0C_0, B_1 on A_0C_0 and C_1 on A_0B_0 such that:

$$\frac{B_0A_1}{A_1C_0} = \frac{C_0B_1}{B_1A_0} = \frac{A_0C_1}{C_1B_0} = \frac{t}{1-t},$$

where t is a fixed real number in $(0, 1)$. Thus, the next triangle of the sequence is $A_1B_1C_1$. By using the fixed division ratio $t : (1 - t)$ we produce the members of the sequence consecutively (see Figure 1 where $t = \frac{1}{3}$).

In [1] a more complicated sequence of triangles is investigated thoroughly. The author uses complex analysis and so the vertices of a triangle can be defined by three complex numbers A_n, B_n, C_n on the complex plane. The basic iterative process that is studied in [1] has the following matrix form:

$$V_n = TV_{n-1}, \tag{1}$$

where $V_n = \begin{pmatrix} A_n \\ B_n \\ C_n \end{pmatrix}$, $T = \begin{pmatrix} 0 & 1-t & t \\ t & 0 & 1-t \\ 1-t & t & 0 \end{pmatrix}$ is a circulant matrix and V_0 is a given initial triangle. Note that in [1] t is considered generally as a complex number. We stress also that throughout the article we ignore the scaling factor $1/r_n$ that appears at the above iteration in [1]. This factor does not affect the shape of the triangles. As an exceptional case in Section 5 in [1], it is studied the above sequence with t a real number in $(0, 1)$. This is exactly the sequence that

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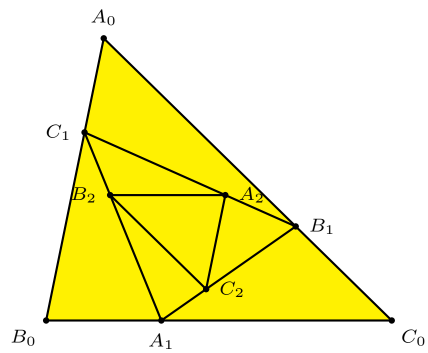

Figure 1.

we described previously and we study in this article. From now on we call this sequence the FDRS (*i.e.*, Fixed Division Ratio Sequence). Concerning the FDRS the author in [1] proved that if

$$t = \frac{1}{2} + \frac{1}{2\sqrt{3}}\tan(a\pi), \tag{2}$$

and a is a rational number, then the FDRS is periodic with respect to the shapes of the triangles. Apparently the same result is proved in [3] (although the proof is left as an exercise). At first sight the formula for the periodicity in [3] seems quite different from (2), but after some algebraic calculations it can be shown that is indeed the same. In [3] it is also proved, that the limiting point of the FDRS is the centroid of the initial triangle $A_0B_0C_0$. Obviously, this is a direct result from the recurrence (1) since it holds $A_{n+1} + B_{n+1} + C_{n+1} = A_n + B_n + C_n$, which means that all the triangles of the FDRS have the same centroid.

In this article we are interested in the behavior of the shapes of the triangles in the FDRS. Particularly, we examine the case when a in (2) is an irrational number. This case was not examined in [1] and [3]. Throughout the article we use the same nomenclature as in [1] and our results are an addendum to [1].

2. Preliminary results

In this Section we will repeat the formulation and the basic results from [1] and we will present some significant remarks. We use the recurrence (1) which is the FDRS as it represented on the complex plane. Without loss of generality as in [1], we can consider that the centroid of the initial triangle $A_0B_0C_0$ is at the origin (*i.e.*, $A_0+B_0+C_0 = 0$). This is legitimate since it is just a translation of the centroid to the origin and it does not affect the shapes of the triangles of the FDRS. By using results from circulant matrix theory in [1], it is proved that

$$V_n = T^n V_0 = s_1\lambda_1^n F_{3,1} + s_2\lambda_2^n F_{3,2} \tag{3}$$

where $F_{3,1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}$ and $F_{3,2} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \omega^2 \\ \omega^4 \end{pmatrix}$ are columns of the 3×3 Fourier matrix $F_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{pmatrix}$. Moreover, $\lambda_j = (1-t)\omega^j + t\omega^{2j}$, $j = 0, 1, 2$ are the eigenvalues of T and $s = \begin{pmatrix} s_0 \\ s_1 \\ s_2 \end{pmatrix}$ such that $F_3 s = V_0$. We also consider $\omega = e^{i2\pi/3}$, $\eta = e^{i\pi/3}$ and as $\overline{x}$ we denote the conjugate of x. The following function $z : \mathbb{C}^3 \to \mathbb{C}$ is also defined in [1]:

$$z(V_n) = \frac{C_n - A_n}{B_n - A_n}. \tag{4}$$

This is a very useful function. First, it signifies the orientation of the triangle on the complex plane. Thus, if $\arg(z(V_n)) > 0$ (< 0) the triangle is positively (negatively) oriented (see Figure 2). Note also the angle $\widehat{A}_n$ of the triangle $A_n B_n C_n$ is equal to $\arg(z(V_n))$, so $\widehat{A}_n$ can be regarded as positive or negative.

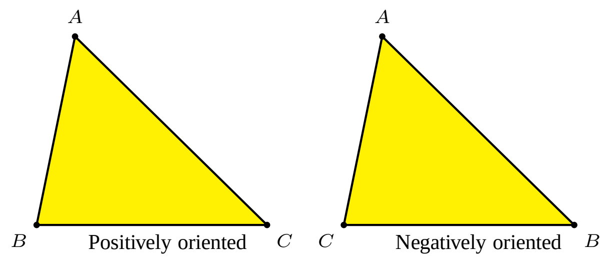

Figure 2.

Function $z(V_n)$ also signifies the ratio of the sides b_n, c_n since $|z(V_n)| = \frac{b_n}{c_n}$. If for instance we have that $|z(V_n)| = 1$, the triangle is isosceles ($b_n = c_n$). If additionally we have $\arg(z(V_n)) = \pi/3$ or $\arg(z(V_n)) = -\pi/3$ then the triangle is equilateral. From this observation we have the following Proposition:

Proposition 1. *A triangle $A_n B_n C_n$ on the complex plane is equilateral if and only if $z(V_n) = \eta$ (positively oriented) or $z(V_n) = \overline{\eta}$ (negatively oriented).*

All these facts stress the importance of function (4). It is apparent that the shape of a triangle on the complex plane is determined completely by function (4). Now, let us assume that the initial triangle $A_0 B_0 C_0$ of the FDRS is not degenerate (*i.e.*, two or three vertices do not coincide and the vertices are not collinear). Moreover, let us assume that $A_0 B_0 C_0$ is not equilateral (*i.e.*, $z(V_0) \neq \eta$ and $z(V_0) \neq \overline{\eta}$), because if it was equilateral then all members of the FDRS would be equilateral triangles. Let us next present two significant definitions and notations.

Firstly, after some algebraic calculations we define the following ratio:

$$\frac{s_2}{s_1} = \frac{B_0 - \omega A_0}{\omega^2 A_0 - B_0} = re^{i\rho}, \tag{5}$$

where $r = |\frac{s_2}{s_1}|$ and $\rho = \arg(\frac{s_2}{s_1})$. Note that (5) holds because we have considered $A_0 + B_0 + C_0 = 0$.

Secondly, from the eigenvalues λ_1 and λ_2 we can get the following definitions

$$\frac{\lambda_2}{\lambda_1} = e^{i\theta}, \quad and \quad \theta = 2\arctan(\sqrt{3}(2t-1)). \tag{6}$$

If we let $\theta = 2\pi a$ in the above equation we get directly equation (2). Now, we can consider the following cases:

(1) $\theta = 0$. In this case we have $t = 1/2$ and all the members of the FDRS are similar to $A_0B_0C_0$.
(2) $\theta = 2k\pi/m$. This case is studied in [1] where $a = k/m$ is rational. We have a periodical behavior and if $(k,m) = 1$ the period is equal to m (otherwise it is smaller than m).
(3) $\theta = 2a\pi$, where a is irrational. This is the case that we study in this article.

In what follows we prove a number of important facts about the FDRS.

Firstly, we note that it holds $s_1 \neq 0$ and $s_2 \neq 0$. This is a straightforward result from the equality $z(V_0) = \frac{s_1\eta + s_2}{s_1 + s_2\eta}$ (see [1]) and from the assumption that $z(V_n) \neq \eta$ and $z(V_n) \neq \overline{\eta}$.

Our next aim is to prove that $r \neq 1$. Let $A_0 = a_1 + ia_2$ and $B_0 = b_1 + ib_2$ and assume that $r = 1$ or equivalently $|B_0 - \omega A_0| = |\omega^2 A_0 - B_0|$. After some algebraic calculations we find $a_1b_2 = a_2b_1$, which means that the determinant $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0$ and so the vectors $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ are linearly dependent. Thus, $A_0 = \lambda B_0$ where λ is real and $\lambda \neq 0$, $\lambda \neq 1$. Now from (4) we get

$$z(V_0) = \frac{C_0 - A_0}{B_0 - A_0} = \frac{-B_0 - 2A_0}{B_0 - A_0} = -\frac{1 + 2\lambda}{1 - \lambda} \in R.$$

Thus, $\arg(z(V_0)) = 0$ or $\arg(z(V_0)) = \pi$ which is impossible since the initial triangle is not degenerate. Consequently, it holds $r \neq 1$.

Next, we examine the case $r < 1$. From (3) and (6) we have

$$V_n = \lambda_1^n(s_1F_{3,1} + s_2e^{in\theta}F_{3,2}).$$

By using the above equation and (5), equation (4) becomes

$$z(V_n) = \frac{s_1\eta + s_2e^{in\theta}}{s_1 + s_2\eta e^{in\theta}} = \eta\frac{1 + re^{i(\varphi_n - \pi/3)}}{1 + re^{i(\varphi_n + \pi/3)}},$$

where $\varphi_n = n\theta + \rho$. From the above equation we get directly that:

$$\begin{aligned} \arg(z(V_n)) = {} & \widehat{A}_n = \Phi(\varphi_n, r) = \\ = {} & \frac{\pi}{3} + \arctan\frac{r\sin(\varphi_n - \pi/3)}{1 + r\cos(\varphi_n - \pi/3)} - \arctan\frac{r\sin(\varphi_n + \pi/3)}{1 + r\cos(\varphi_n + \pi/3)}, \end{aligned} \tag{7}$$

and

$$|z(V_n)| = \frac{b_n}{c_n} = \mu(\varphi_n, r) = \tag{8}$$
$$= \sqrt{\frac{(1 + r\cos(\varphi_n - \pi/3))^2 + r^2\sin^2(\varphi_n - \pi/3)}{(1 + r\cos(\varphi_n + \pi/3))^2 + r^2\sin^2(\varphi_n + \pi/3)}}.$$

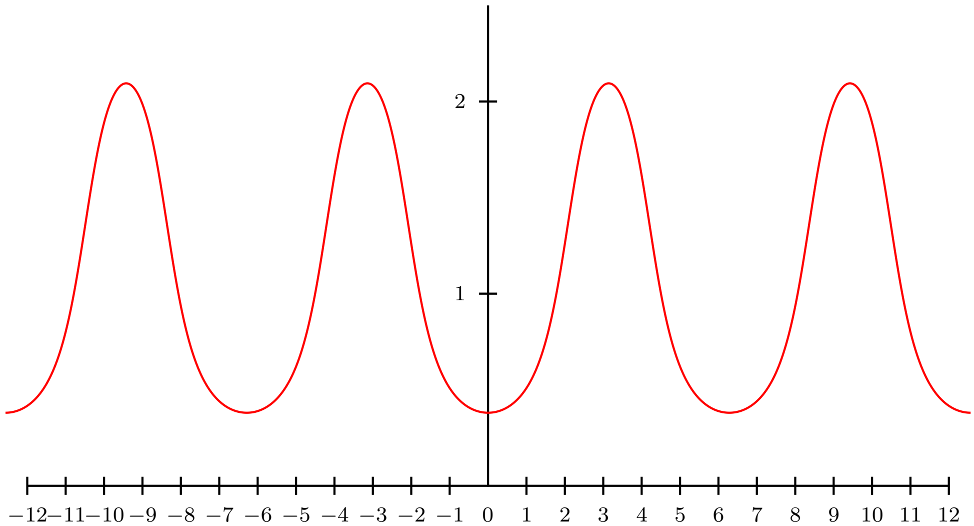

Figure 3(a)

Observe that in (7) and (8) functions $\Phi(\varphi, r)$ and $\mu(\varphi, r)$ are defined respectively. We also define $\Phi(\varphi) = \Phi(\varphi, r)$ and $\mu(\varphi) = \mu(\varphi, r)$. Function $\Phi(\varphi)$ is even (*i.e.*, $\Phi(\varphi) = \Phi(-\varphi)$) and periodic with period 2π (see Figure 3(a) where $r = 0.5$). The minima of $\Phi(\varphi)$ appear at $\varphi = 0, \pm 2\pi, \pm 4\pi, \ldots$ and the maxima at $\varphi = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$. Thus, $\arg(z(V_n)) = \Phi(\varphi_n, r) \in [m_1, m_2]$ where

$$m_1 = \Phi(0, r) = \frac{\pi}{3} - 2\arctan\frac{r\sqrt{3}}{2 + r}, \qquad m_2 = \Phi(\pi, r) = \frac{\pi}{3} + 2\arctan\frac{r\sqrt{3}}{2 - r}.$$

In Figure 3(b), where function Φ is depicted for different values of r, we can observe that the interval $[m_1, m_2]$ decreases as $r \to 0^+$ and increases as $r \to 1^-$. In every case since $r \in (0, 1)$ we find that $[m_1, m_2] \subset (0, \pi)$, which also means that the triangles of the FDRS are positively oriented.

Concerning function $\mu(\varphi)$ we have the following properties: $\mu(k\pi) = 1$ where k is integer, $\mu(-\varphi) = 1/\mu(\varphi)$ and $\mu(\varphi)$ is periodic with period 2π. Figure 3(c) depicts function $\mu(\varphi)$ in $[-4\pi, 4\pi]$ and $r = 0.5$.

Remark. Let us present a fact that we will need in Section 3. Let $r < 1$, since a similar argument applies for $r > 1$. Recall that function $\Phi(\varphi)$ is not injective (one-to-one) and so its inverse can not be determined uniquely. For an angle $\tilde{\theta} \in [m_1, m_2]$ (*i.e.*, $\tilde{\theta}$ belongs to the range of Φ), we want to find the elements $\tilde{\varphi}_m$

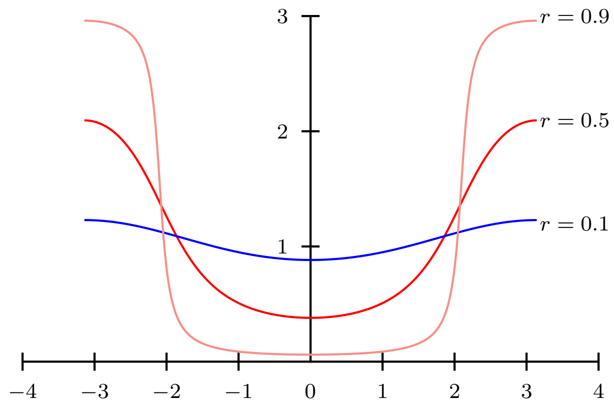

Figure 3(b)

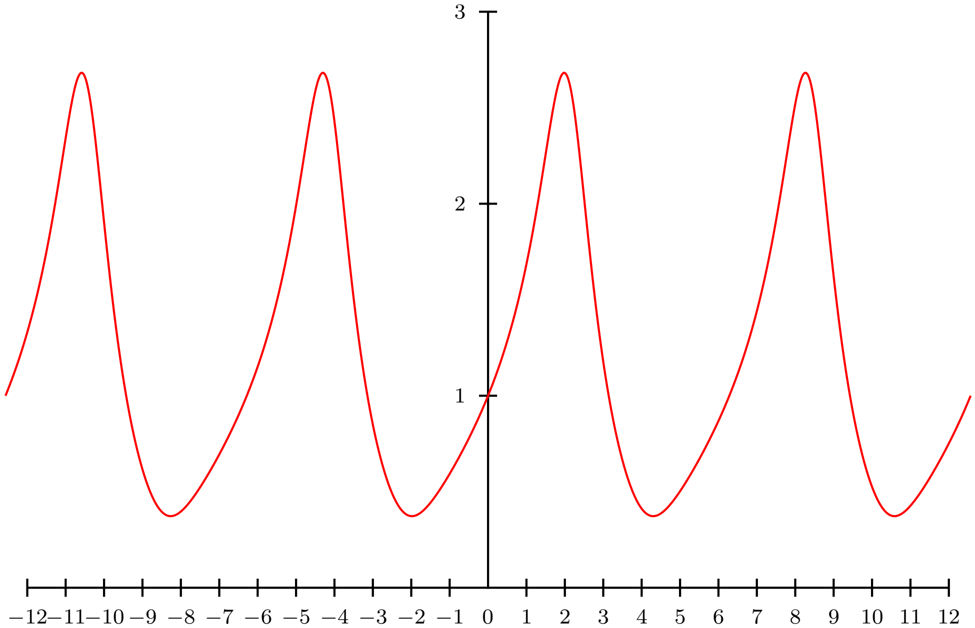

Figure 3(c)

which have the same image $\tilde{\theta}$ (*i.e.*, $\Phi(\tilde{\varphi}_m) = \tilde{\theta}$). Since $\Phi(\varphi)$ is periodic with period 2π, the elements $\tilde{\varphi}_m$ have the form: $2k\pi \pm \varphi_a(\tilde{\theta})$ (k is integer), where as $\varphi_a(\tilde{\theta})$ we define the minimum element $\tilde{\varphi}_m$ such that $\tilde{\varphi}_m \geq 0$ (see Figure 4). Apparently, $\varphi_a(\tilde{\theta}) \in [0, \pi]$ and it holds that $\Phi(2k\pi \pm \varphi_a(\tilde{\theta})) = \tilde{\theta}$ (*i.e.*, all the elements $2k\pi \pm \varphi_a(\tilde{\theta})$ have the same image $\tilde{\theta}$). Figure 4 depicts this characteristic of function $\Phi(\varphi)$.

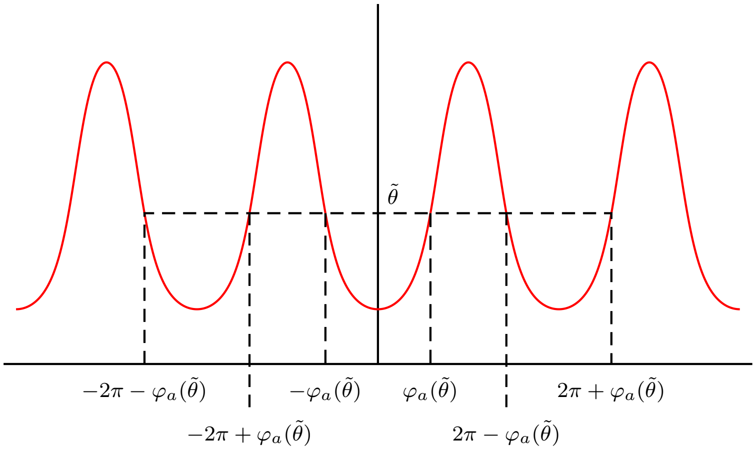

Figure 4.

Now, for the case $|\frac{s_2}{s_1}| = r > 1$ we can use the inverse ratios $\frac{s_1}{s_2} = \frac{1}{r}e^{-i\rho}$, $\frac{\lambda_1}{\lambda_2} = e^{-i\theta}$ and have that

$$z(V_n) = \frac{s_1\eta + s_2 e^{in\theta}}{s_1 + s_2\eta e^{in\theta}} = \overline{\eta}\frac{1+\frac{1}{r}e^{i(-\varphi_n+\pi/3)}}{1+\frac{1}{r}e^{i(-\varphi_n-\pi/3)}},$$

where again $\varphi_n = n\theta+\rho$. From the above we have as before:

$$\begin{aligned}\arg(z(V_n)) &= \widehat{A}_n = -\Phi(-\varphi_n, 1/r) = \\ &= -\frac{\pi}{3} + \arctan\frac{\frac{1}{r}\sin(-\varphi_n+\pi/3)}{1+\frac{1}{r}\cos(-\varphi_n+\pi/3)} - \arctan\frac{\frac{1}{r}\sin(-\varphi_n-\pi/3)}{1+\frac{1}{r}\cos(-\varphi_n-\pi/3)},\end{aligned} \tag{9}$$

and

$$\begin{aligned}|z(V_n)| &= \frac{b_n}{c_n} = \frac{1}{\mu(-\varphi_n, 1/r)} = \\ &= \sqrt{\frac{(1+\frac{1}{r}\cos(-\varphi_n+\pi/3))^2 + \frac{1}{r^2}\sin^2(-\varphi_n+\pi/3)}{(1+\frac{1}{r}\cos(-\varphi_n-\pi/3))^2 + \frac{1}{r^2}\sin^2(-\varphi_n-\pi/3)}}.\end{aligned} \tag{10}$$

It is now obvious that equations (7), (8) and equations (9), (10) signify similar triangles with different orientations provided of course that φ_n and r are common. When $r > 1$ the triangles of the FDRS are negatively oriented. Using similar arguments as before we can prove easily that $\arg(z(V_n)) = \widehat{A}_n \in [\overline{m}_1, \overline{m}_2]$ where

$$\overline{m}_1 = -\Phi(-\pi, 1/r) = -\frac{\pi}{3} - 2\arctan\frac{\frac{1}{r}\sqrt{3}}{2-\frac{1}{r}},$$

$$\overline{m}_2 = -\Phi(0, 1/r) = -\frac{\pi}{3} + 2\arctan\frac{\frac{1}{r}\sqrt{3}}{2+\frac{1}{r}}.$$

Thus, for any $r > 1$ we have $[\overline{m}_1, \overline{m}_2] \subset (-\pi, 0)$. The interval $[\overline{m}_1, \overline{m}_2]$ increases as $r \to 1^+$ and decreases as $r \to +\infty$. In the next Section we apply Kronecker's Approximation Theorem in order to get our main result for the FDRS when a in (2) is an irrational number.

3. Application of Kronecker's approximation theorem

First we present Kronecker's Approximation Theorem (see e.g. [4]).

Kronecker's approximation theorem *If ω is a given irrational number, then the sequence of numbers $\{n\omega\}$, where $\{x\} = x - \lfloor x \rfloor$, is dense in the unit interval. Explicitly, given any $\overline{p}$, $0 \leq \overline{p} \leq 1$, and given any $\epsilon > 0$, there exists a positive integer k such that $|\{k\omega\} - \overline{p}| < \epsilon$.*

We know that $\varphi_n = n\theta + \rho = 2\pi a n + \rho$ and recall that a is irrational and ρ is a function of A_0, B_0, so it is fixed. From Kronecker's Approximation Theorem we know that a member of the sequence $\{na\} = na - \lfloor na \rfloor$ will be arbitrarily close to any given $\overline{p} \in [0, 1]$. Similarly, a member of the sequence $2\pi\{na\} = \varphi_n - 2\pi \lfloor na \rfloor - \rho$ will be arbitrarily close to the angle $\overline{\theta} = 2\pi\overline{p} \in [0, 2\pi]$. Thus, a member of the sequence φ_n will be arbitrarily close to the angle $\overline{\theta} + 2\pi \lfloor na \rfloor + \rho$. Let us now define the sequence of angles φ_n on the unit circle. The quantity $2\pi \lfloor na \rfloor$ defines complete rotations on the unit circle and can be eliminated. This implies that a member of the sequence φ_n will be arbitrarily close to the angle $\overline{\theta} + \rho$ on the unit circle. If additionally, we imagine the unit circle to rotate by $-\rho$, we get that a member of the sequence φ_n will be arbitrarily close to the angle $\overline{\theta} = 2\pi\overline{p}$ on the unit circle. Since this holds for any given $\overline{p} \in [0, 1]$, we conclude that a member of the sequence φ_n will be arbitrarily close to any given angle $\overline{\theta} \in [0, 2\pi]$ on the unit circle. This important fact will be used in the proof of the next Theorem which is the main result of this article. Note that the Theorem uses the notation that has already been presented.

Theorem 2. *Let A_0, B_0, C_0 be complex numbers which define an initial non-degenerate and non-equilateral triangle on the complex plane such that its centroid is at the origin (*i.e.*, $A_0 + B_0 + C_0 = 0$). Suppose we apply the FDRS with $t = \frac{1}{2} + \frac{1}{2\sqrt{3}} \tan(a\pi)$ where a is an irrational number. Let $\epsilon_1 > 0$ and $\epsilon_2 > 0$. We have the following cases:*

(1) If $r = |\frac{s_2}{s_1}| < 1$ (positively oriented triangles), choose a $\widetilde{\theta} \in [m_1, m_2] \subset (0, \pi)$. Then there is a member of the FDRS $A_k B_k C_k$ such that:

$$|\widehat{A}_k - \widetilde{\theta}| < \epsilon_1,$$

and

$$either \quad \left| \frac{b_k}{c_k} - \mu(\varphi_a(\widetilde{\theta}), r) \right| < \epsilon_2 \quad or \quad \left| \frac{b_k}{c_k} - \mu^{-1}(\varphi_a(\widetilde{\theta}), r) \right| < \epsilon_2.$$

(2) If $r = |\frac{s_2}{s_1}| > 1$ (negatively oriented triangles), choose a $\widetilde{\theta} \in [\overline{m}_1, \overline{m}_2] \subset (-\pi, 0)$. Then there is a member of the FDRS $A_k B_k C_k$ such that:

$$|\widehat{A}_k - \widetilde{\theta}| < \epsilon_1,$$

and

$$either \quad \left|\frac{b_k}{c_k} - \mu(\varphi_a(\widetilde{\theta}), 1/r)\right| < \epsilon_2 \quad or \quad \left|\frac{b_k}{c_k} - \mu^{-1}(\varphi_a(\widetilde{\theta}), 1/r)\right| < \epsilon_2.$$

Proof: Let $r < 1$, we have seen that there is φ_k which is arbitrarily close to any given angle on the unit circle. Since function $\Phi(\varphi_n)$ is continuous with respect to φ_n, it is apparent that $\widehat{A}_k = \Phi(\varphi_k)$ can be arbitrarily close to a $\widetilde{\theta}$ chosen from the interval $[m_1, m_2]$ (the range of $\Phi(\varphi_n)$). This proves that $|\widehat{A}_k - \widetilde{\theta}| < \epsilon_1$. Since $\widehat{A}_k = \Phi(\varphi_k)$ can be arbitrarily close to $\widetilde{\theta}$, from Remark we conclude that φ_k will be arbitrarily close to an element of the form $2k\pi \pm \varphi_a(\tilde{\theta})$ (see Figure 4). Since we have considered that φ_k can be defined on the unit circle, we have that φ_k will be arbitrarily close either to $\varphi_a(\tilde{\theta})$ or to $2\pi - \varphi_a(\tilde{\theta})$ which are both defined in $[0, 2\pi]$. Observe that function $\mu(\varphi_n, r)$ is continuous with respect to φ_n and so from equation (8) we get that the ratio $\frac{b_k}{c_k} = \mu(\varphi_k, r)$ will be arbitrarily close either to $\mu(\varphi_a(\widetilde{\theta}), r)$ or to $\mu(2\pi - \varphi_a(\widetilde{\theta}), r) = \mu(-\varphi_a(\widetilde{\theta}), r) = \mu^{-1}(\varphi_a(\widetilde{\theta}), r)$ (recall the properties of function μ). This proves that either $\left|\frac{b_k}{c_k} - \mu(\varphi_a(\widetilde{\theta}), r)\right| < \epsilon_2$ or $\left|\frac{b_k}{c_k} - \mu^{-1}(\varphi_a(\widetilde{\theta}), r)\right| < \epsilon_2$. The case $r > 1$ can be treated analogously. This completes the proof. □

Concerning Theorem 2 we stress that ϵ_1 and ϵ_2 can be chosen independently. This is true since from the Kronecker's Approximation Theorem we can always find a φ_k as close as we want to a given $\widetilde{\theta}$. This implies that the angle $\widehat{A}_k$ can be as close as we want to $\widetilde{\theta}$, and so the ratio $\frac{b_k}{c_k}$ will be as close as we want either to $\mu(\varphi_a(\widetilde{\theta}), r)$ or to $\mu^{-1}(\varphi_a(\widetilde{\theta}), r)$. Ultimately, a φ_k will satisfy both inequalities no matter how small ϵ_1 and ϵ_2 are.

Although Theorem 2 and the analysis so far seem quite complicated, they have some interesting consequences. In what follows we consider that t is fixed and a is an irrational number as in Theorem 2.

We proved that there will be a member of the FDRS with an angle $\widehat{A}_k$ that will be arbitrarily close to any given $\widetilde{\theta} \in [m_1, m_2]$ or $\widetilde{\theta} \in [\overline{m}_1, \overline{m}_2]$. This means that the countable set of the angles $\widehat{A}_n$ (*i.e.*, $\{\widehat{A}_0, \widehat{A}_1, \dots\}$) is dense in $[m_1, m_2]$ or in $[\overline{m}_1, \overline{m}_2]$. Also by choosing ϵ_1, ϵ_2 as small as we want, we expect that some members $A_k B_k C_k$ of the FDRS will have their shapes as follows: $\widehat{A}_k \simeq \widetilde{\theta}$ and either $\frac{b_k}{c_k} \simeq \mu(\varphi_a(\widetilde{\theta}), r)$ or $\frac{b_k}{c_k} \simeq \mu^{-1}(\varphi_a(\widetilde{\theta}), r)$.

Let us now find if there is a member of the FDRS that is arbitrarily close to an equilateral triangle. If this was true then $\frac{b_k}{c_k}$ should be arbitrarily close to the unity. Thus from Theorem 2 (assume that $r < 1$ since for $r > 1$ the same argument applies), $\mu(\varphi_a(\widetilde{\theta}), r) = 1$ and from Section 2 we know that $\varphi_a(\widetilde{\theta}) = 0$ or $\varphi_a(\widetilde{\theta}) = \pi$. From these equalities we get $\widetilde{\theta} = m_1$ or $\widetilde{\theta} = m_2$. It should also hold that $\widetilde{\theta} = \pi/3$ (positively oriented equilateral triangle). So, it should be $m_1 = \pi/3 \Rightarrow r = 0$ or $m_2 = \pi/3 \Rightarrow r = 0$. Obviously, $r = 0$ is impossible. Consequently, for a specific $r > 0$ all the members of the FDRS will have at least a constant discrepancy from the shape of an equilateral triangle. This discrepancy can not be further decreased for a fixed $r > 0$, it can only be reduced if we chose another $r > 0$ closer to zero.

Let an isosceles triangle with $b = c$ and $\widehat{A} = \widetilde{\theta} < \pi/3$ be given. We want to find the value of $r < 1$ that will give a member of the FDRS arbitrarily close to the isosceles triangle. In the previous paragraph we show that for this case it holds $\widetilde{\theta} = m_1$ or $\widetilde{\theta} = m_2$. Let $\widetilde{\theta} = m_1$ and we have

$$\widetilde{\theta} = m_1 \Longleftrightarrow 2\arctan\frac{r\sqrt{3}}{2+r} = \frac{\pi}{3} - \widetilde{\theta} \Longleftrightarrow r = \frac{2\tan(\frac{\pi}{6} - \frac{\widetilde{\theta}}{2})}{\sqrt{3} - \tan(\frac{\pi}{6} - \frac{\widetilde{\theta}}{2})}.$$

The above formula gives the value of r for which a member of the FDRS would be arbitrarily close to the isosceles triangle with $\widehat{A} = \widetilde{\theta} < \pi/3$. The corresponding formula for an isosceles triangle with $b = c$ and a given $\widehat{A} = \widetilde{\theta} > \pi/3$ is

$$\widetilde{\theta} = m_2 \Longleftrightarrow 2\arctan\frac{r\sqrt{3}}{2-r} = \widetilde{\theta} - \frac{\pi}{3} \Longleftrightarrow r = \frac{2\tan(\frac{\widetilde{\theta}}{2} - \frac{\pi}{6})}{\sqrt{3} + \tan(\frac{\widetilde{\theta}}{2} - \frac{\pi}{6})}.$$

In the next Section we offer a simple geometric presentation of the FDRS, we examine closer the significance of the parameters r and φ_n and we answer a question posed in [3].

4. Geometric interpretations and final remarks

We have seen that equation (3) is the solution of the recurrence (1) provided that $A_0 + B_0 + C_0 = 0$. We can rewrite (3) as follows:

$$V_n = \frac{s_1\lambda_1^n}{\sqrt{3}}\left[\begin{pmatrix}1\\ \omega\\ \omega^2\end{pmatrix} + \frac{s_2}{s_1}\left(\frac{\lambda_2}{\lambda_1}\right)^n\begin{pmatrix}1\\ \omega^2\\ \omega\end{pmatrix}\right].$$

In this article we are interested in the shapes of the triangles. The complex number $\frac{s_1\lambda_1^n}{\sqrt{3}}$ at the above equation signifies a scaling factor and a rotation of the triangle V_n, and so it does not affect its shape. This means that we can define the shapes of the triangles of the FDRS simply as

$$S_n = P + re^{i\varphi_n}N, \tag{11}$$

where $P = \begin{pmatrix}1\\ \omega\\ \omega^2\end{pmatrix}$, $N = \begin{pmatrix}1\\ \omega^2\\ \omega\end{pmatrix}$ and r, φ_n as in Section 2. We stress that the triangles V_n and S_n have the same shape (*i.e.*, they are similar and they have the same orientation). Note also that P is a positively oriented equilateral triangle inscribed in the unit circle and N is a negatively oriented equilateral triangle inscribed in the unit circle (1, ω, ω^2 are the third roots of unity). It can be seen now that every member of the FDRS on the complex plane is represented as the sum of two equilateral triangles: P and $re^{i\varphi_n}N$. It is now obvious that the parameter r is the circumradius and the parameter φ_n is the angle of rotation of the equilateral triangle rN at the nth iteration. Thus, the parameters r and φ_n determine completely the contribution of the negatively oriented triangle in (11).

Let us next consider an open problem that is posed in [3]. The authors of [3] asked to find all values of the division ratio $t \in (0,1)$ for which the FDRS is divergent in shape. From the analysis so far, we have seen that the division ratio t can be given by equation (2). Equation (2) defines a function $t = t(a)$ which is one-to-one and for $a \in (-\frac{1}{3}, \frac{1}{3})$ its range is $(0,1)$. Thus, we can describe the behavior of the members of the FDRS with respect to t, by using equation (2). Similar to the analysis of Section 2 we have the following cases:

(1) $a = 0$. Equation (2) implies $t = \frac{1}{2}$. In this case all the members of the FDRS are similar to $A_0B_0C_0$ and the sequence is convergent in shape.
(2) $a \neq 0$ is a rational number in $(-\frac{1}{3}, \frac{1}{3})$ and t is given by (2). The FDRS is periodic in shape.
(3) a is an irrational number in $(-\frac{1}{3}, \frac{1}{3})$ and t is given by (2). From the analysis of Section 3 we conclude that the FDRS is neither convergent nor periodic in shape.

Thus, only when $t = \frac{1}{2}$ we have that the FDRS is convergent in shape. The second case above gives the values of t for which the FDRS is periodic in shape. The last case is described by Theorem 2 and the behavior of the FDRS is rather complex since it is neither convergent nor periodic in shape.

It is clear that only the change of an a rational to an a irrational in (2) is enough to produce a complicated dynamic behavior of the FDRS. We believe that only results of qualitative character like Theorem 2 can be used to describe this sequence of triangles. However, it would be interesting if one could prove another result (e.g. a statistical result), for the behavior of the FDRS when a is an irrational number.

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