Forum Geometricorum
Volume 8 (2008) 61–62.

FORUM GEOM
ISSN 1534-1178

A Simple Ruler and Rusty Compass Construction of the Regular Pentagon

Kurt Hofstetter

Abstract. We construct in 13 steps a regular pentagon with given sidelength using a ruler and rusty compass.

Suppose a line segment AB has been divided in the golden ratio at a point G. Figure 1 shows the construction of the vertices of a regular pentagon with four circles of radii equal to AB. Thus, let $\mathcal{C}_1 = A(AB)$, $\mathcal{C}_2 = B(AB)$, $\mathcal{C}_4 = G(AB)$, intersecting the half line AB at P_1, and $\mathcal{C}_5 = P_1(AB)$. Then, with $P_2 = \mathcal{C}_1 \cap \mathcal{C}_5$, $P_4 = \mathcal{C}_1 \cap \mathcal{C}_4$, and $P_5 = \mathcal{C}_2 \cap \mathcal{C}_5$. Since the radii of the circles involved are equal, this construction can be performed with a ruler and a rusty compass. We claim that the pentagon $P_1P_2AP_4P_5$ is regular.

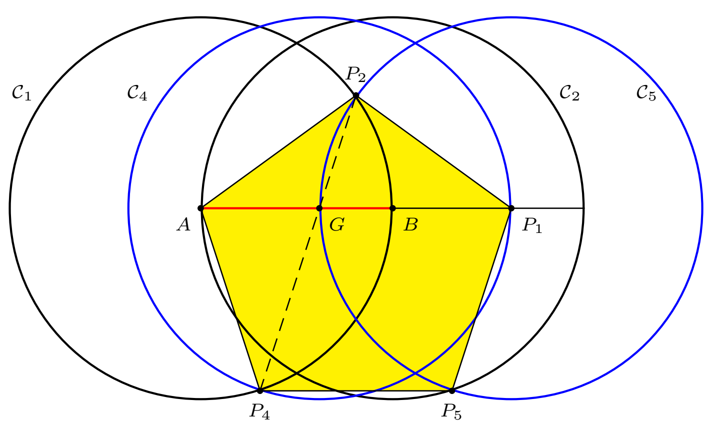

Figure 1

Here is a simple proof. Assume unit length for the segment AB. Let $\phi := \frac{\sqrt{5}+1}{2}$ be the golden ratio. It is well known that $AG = \frac{1}{\phi} = \phi - 1$. Now, $AP_1 = (\phi - 1) + 1 = \phi$. Therefore, the isosceles triangle AP_1P_2 consists of two sides and a diagonal of a regular pentagon. In particular, $\angle P_2AP_1 = 36°$ and $\angle AP_2P_1 = 108°$. On the other hand, triangle AGP_4 is also isosceles with sides in the proportions $1 : 1 : \frac{1}{\phi} = \phi : \phi : 1$. It consists of two diagonals and a side of a regular pentagon. In particular, $\angle GAP_4 = 72°$ and $\angle AP_4G = 36°$. From these, $\angle P_2AP_4 = 36° + 72° = 108°$.

Publication Date: March 24, 2008. Communicating Editor: Paul Yiu.

Now, triangles AP_4P_2 and $P_2AP_1$are congruent. It follows that $\angle AP_2P_4 = 36^\circ$, and P_2, G, P_4 are collinear.

By symmetry, we also have $\angle P_2P_1P_5 = 108^\circ$.

In the pentagon $P_1P_2AP_4P_5$, since the angles at P_1, P_2, A are all 108°, those at P_4 and P_5 are also 108°. On the other hand, since the circles $\mathcal{C}_2$ and $\mathcal{C}_5$ are the translations of $\mathcal{C}_1$ and $\mathcal{C}_4$ by the vector $\overrightarrow{AB}$, P_4P_5 has unit length. This shows that the pentagon $P_1P_2AP_4P_5$ is regular.

Now, using a rusty compass (set at a radius equal to AB) we have constructed in [1] the point G in 5 steps, which include the circles $\mathcal{C}_1$ and $\mathcal{C}_2$. (In Figure 2, M is the midpoint of AB, $\mathcal{C}_3 = M(AB)$ intersects $\mathcal{C}_2$ at E on the opposite side of C; $G = CE \cap AB$). It follows that the vertices of the regular pentagon $P_1P_2AP_4P_5$ can be constructed in $5 + 3 = 8$ steps. The pentagon can be completed in 5 more steps by filling in the sides.

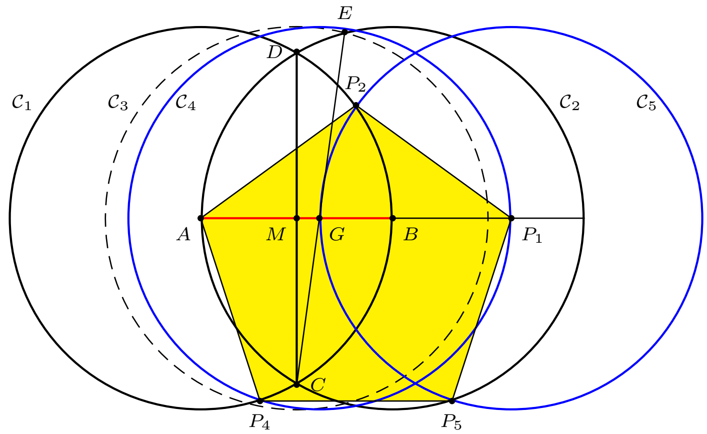

Figure 2

References

[1] K. Hofstetter, Divison of a segment in the golden section with ruler and rusty compass, *Forum Geom.*, 5 (2005) 135–136.

Kurt Hofstetter: Object Hofstetter, Media Art Studio, Langegasse 42/8c, A-1080 Vienna, Austria
E-mail address: hofstetter@sunpendulum.at