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An Inequality Involving the Angle Bisectors and an Interior Point of a Triangle

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Abstract. We establish a new weighted geometric inequality involving the lengths of the angle bisectors and the radii of three circles through an interior point of a triangle. From this, several interesting geometric inequalities are derived.

1. Introduction

Throughout this paper we consider a triangle ABC with sidelengths a, b, c, circumradius R, and inradius r. Denote by w_a, w_b, w_c the lengths of the bisectors of angles A, B, C. Let P be an interior point. Denote by R_a, R_b, R_c the radii of the circles PBC, PCA, PAB respectively. Liu [2] has conjectured the inequality

$$\frac{w_a}{R_b+R_c}+\frac{w_b}{R_c+R_a}+\frac{w_c}{R_a+R_b}\leq\frac{9}{4}. \tag{1}$$

We prove a stronger inequality in Theorem 1 below, which include the

$$\frac{w_a}{\sqrt{R_bR_c}}+\frac{w_b}{\sqrt{R_cR_a}}+\frac{w_c}{\sqrt{R_aR_b}}\leq\frac{9}{2}. \tag{2}$$

Theorem 1. *For an interior point P and positive real numbers x, y, z, we have*

$$\frac{xw_a}{\sqrt{R_bR_c}}+\frac{yw_b}{\sqrt{R_cR_a}}+\frac{zw_c}{\sqrt{R_aR_b}}\leq\sqrt{2+\frac{r}{2R}}\left(\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\right). \tag{3}$$

Equality holds if and only if the triangle ABC is equilateral, P its center, and $x=y=z$.

We shall make use of the following lemma.

Lemma 2. *For arbitrary nonzero real numbers x, y, z,*

$$x^2\sin^2A+y^2\sin^2B+z^2\sin^2C\leq\frac{1}{4}\left(\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\right)^2. \tag{4}$$

Equality holds if and only if $x^2:y^2:z^2=\frac{1}{a^2(b^2+c^2-a^2)}:\frac{1}{b^2(c^2+a^2-b^2)}:\frac{1}{c^2(a^2+b^2-c^2)}$.

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Proof. We make use of Kooi's inequality [1, Inequality 14.1]: for real numbers λ_1, λ_2, λ_3 with $\lambda_1+\lambda_2+\lambda_3 \neq 0$,

$$(\lambda_1+\lambda_2+\lambda_3)^2R^2 \geq \lambda_2\lambda_3a^2+\lambda_3\lambda_1b^2+\lambda_1\lambda_2c^2;$$

equality holds if and only if the point with homogeneous barycentric coordinates $(\lambda_1:\lambda_2:\lambda_3)$ with reference to triangle ABC is the circumcenter of the triangle. Now, with $\lambda_1=\frac{yz}{x}$, $\lambda_2=\frac{zx}{y}$, $\lambda_3=\frac{xy}{z}$, the result follows from the law of sines: $a=2R\sin A$, $b=2R\sin B$, $c=2R\sin C$. □

2. Proof of Theorem 1

The length of the bisector of angle A is given by $w_a=\frac{2bc}{b+c}\cos\frac{A}{2}$. Clearly, $w_a\leq\sqrt{bc}\cos\frac{A}{2}$; equality holds if and only if $b=c$.

Let $\angle BPC=\alpha$, $\angle CPA=\beta$ and $\angle APB=\gamma$. Obviously, $0<\alpha,\ \beta,\ \gamma<\pi$ and $\alpha+\beta+\gamma=2\pi$. By the law of sines, $b=2R_b\sin\beta$, $c=2R_c\sin\gamma$. We have

$$\begin{aligned}\frac{w_a}{\sqrt{R_bR_c}} &\leq \sqrt{\frac{bc}{R_bR_c}}\cdot\cos\frac{A}{2}\\ &=2\sqrt{\sin\beta\sin\gamma}\cdot\cos\frac{A}{2}\\ &\leq(\sin\beta+\sin\gamma)\cos\frac{A}{2}\\ &=2\sin\frac{\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2}\cos\frac{A}{2}\\ &\leq 2\sin\frac{\alpha}{2}\cos\frac{A}{2}.\end{aligned}$$

Equality holds if and only if $b=c$ and $\beta=\gamma$. Similarly, $\frac{w_b}{\sqrt{R_cR_a}}\leq 2\sin\frac{\beta}{2}\cos\frac{B}{2}$ and $\frac{w_c}{\sqrt{R_aR_b}}\leq 2\sin\frac{\gamma}{2}\cos\frac{C}{2}$ with analogous conditions for equality. Therefore, for $x,\ y,\ z>0$,

$$\begin{aligned}&\frac{xw_a}{\sqrt{R_bR_c}}+\frac{yw_b}{\sqrt{R_cR_a}}+\frac{zw_c}{\sqrt{R_aR_b}}\\ &\leq 2x\sin\frac{\alpha}{2}\cos\frac{A}{2}+2y\sin\frac{\beta}{2}\cos\frac{B}{2}+2z\sin\frac{\gamma}{2}\cos\frac{C}{2} \qquad (5)\\ &\leq 2\sqrt{\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)\left(x^2\sin^2\frac{\alpha}{2}+y^2\sin^2\frac{\beta}{2}+z^2\sin^2\frac{\gamma}{2}\right)} \qquad (6)\\ &\leq\sqrt{2+\frac{r}{2R}}\cdot\left(\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\right) \qquad (7)\end{aligned}$$

Here, the inequality in (6) follows from the Cauchy-Schwarz inequality. On the other hand, the inequality in (7) follows from the identity

$$\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2+\frac{r}{2R},$$

and application of Lemma 2 to a triangle with angles $\frac{\alpha}{2}$, $\frac{\beta}{2}$, $\frac{\gamma}{2}$. Equality holds in (5) holds if and only if $a = b = c$ and $\alpha = \beta = \gamma$. This means that ABC is equilateral and P is its center. Finally, by Lemma 2 again, equality holds in (7) if and only if $x^2 : y^2 : z^2 = 1 : 1 : 1$, *i.e.*, $x = y = z$. This completes the proof of Theorem 1.

3. Some applications

With $x = y = z$ in Theorem 1, we have

$$\frac{w_a}{\sqrt{R_b R_c}} + \frac{w_b}{\sqrt{R_c R_a}} + \frac{w_c}{\sqrt{R_a R_b}} \leq 3\sqrt{2 + \frac{r}{2R}}.$$

By Euler's famous inequality $R \geq 2r$, we have (2).

Since $\sqrt{R_b R_c} \leq \frac{1}{2}(R_b + R_c)$, $\sqrt{R_c R_a} \leq \frac{1}{2}(R_c + R_a)$, $\sqrt{R_a R_b} \leq \frac{1}{2}(R_a + R_b)$, we obtain from Theorem 1,

$$\frac{xw_a}{R_b + R_c} + \frac{yw_b}{R_c + R_a} + \frac{zw_c}{R_a + R_b} \leq \frac{1}{2}\sqrt{2 + \frac{r}{2R}}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right). \tag{8}$$

With $x = y = z$, we have

$$\frac{w_a}{R_b + R_c} + \frac{w_b}{R_c + R_a} + \frac{w_c}{R_a + R_b} \leq \frac{3}{2}\sqrt{2 + \frac{r}{2R}}.$$

Liu's inequality (1) follows from $R \geq 2r$.

Again, from Euler's inequality, we immediately conclude from Theorem 1 that

$$\frac{xw_a}{\sqrt{R_b R_c}} + \frac{yw_b}{\sqrt{R_c R_a}} + \frac{zw_c}{\sqrt{R_a R_b}} \leq \frac{3}{2}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right). \tag{9}$$

Corollary 3. *For an interior point P and positive real numbers x, y, z, we have*

$$x^2 R_a + y^2 R_b + z^2 R_c \geq \frac{2}{3}(yzw_a + zxw_b + xyw_c).$$

Equality holds if and only if the triangle ABC is equilateral, P its center, and $x = y = z$.

Proof. Replace in (9) x, y, z respectively by $yz\sqrt{R_b R_c}$, $zx\sqrt{R_c R_a}$, $xy\sqrt{R_a R_b}$. □

In particular, with $x = y = z = 1$, we have

$$R_a + R_b + R_c \geq \frac{2}{3}(w_a + w_b + w_c);$$

equality holds if and only if the triangle is equilateral and P its center.

Corollary 4. *For an interior point P in a triangle ABC, $R_a R_b R_c \geq \frac{64}{27} w_a w_b w_c$. Equality holds if and only if ABC is equilateral and P its center.*

Proof. This follows from (9) by putting $x = y = z$ and applying the AM-GM inequality. □

References

[1] O. Bottema et al, *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969.
[2] J. Liu, A hundred unsolved triangle inequality problems, in *Geometric Inequalities in China* (in Chinese), Jiangsu Education Press, Nanjing, 1996.

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