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A Short Proof of Lemoine's Theorem

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Abstract. We give a short proof of Lemoine's theorem that the Lemoine point of a triangle is the unique point which is the centroid of its own pedal triangle.

Lemoine's theorem states that the Lemoine (symmedian) point of a triangle is the unique point which is the centroid of its own pedal triangle. A proof of the fact that the Lemoine point has this property can be found in Honsberger [4, p.72]. The uniqueness part was conjectured by Clark Kimberling in the very first Hyacinthos message [6], and was subsequently confirmed by computations by Barry Wolk [7], Jean-Pierre Ehrmann [2], and Paul Yiu [8, §4.6.2]. Darij Grinberg [3] has given a synthetic proof. In this note we give a short proof by applying two elegant results on orthologic triangles.

Lemma 1. *If P is a point in plane of triangle ABC, with pedal triangle $A'B'C'$, then the perpendiculars from A to $B'C'$, from B to $C'A'$, from C to $A'B'$ are concurrent at Q, the isogonal conjugate of P.*

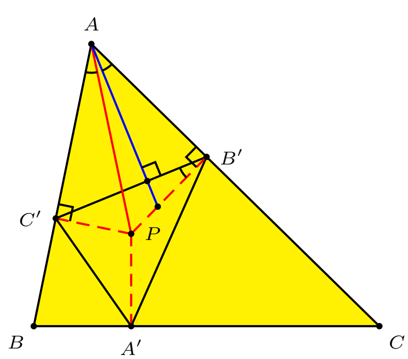

Figure 1

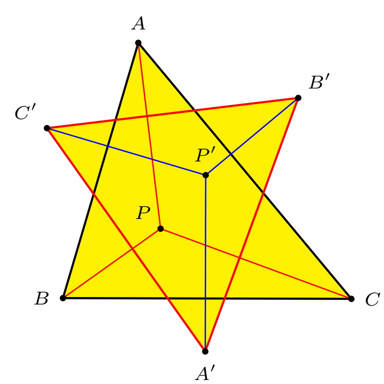

Figure 2

This is quite well-known. See, for example, [5, Theorem 237]. Figure 1 shows that AP and the perpendicular from A to $B'C'$ are isogonal with reference to A. From this Lemma 1 follows. The next beautiful result, illustrated in Figure 2, is the main subject of [1].

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Theorem 2 (Daneels and Dergiades)**.** *If triangles ABC and $A'B'C'$ are orthologic with centers P, P', with the perpendiculars from A, B, C to $B'C'$, $C'A'$, $A'B'$ intersecting at P and those from A', B', C' to BC, CA, AB intersecting at P', then the barycentric coordinates of P with reference to ABC are equal to the barycentric coordinates of P' with reference to $A'B'C'$.*

Now we prove Lemoine's theorem.

Let K be the Lemoine (symmedian) point of triangle ABC, and $A'B'C'$ its pedal triangle. According to Lemma 1, the perpendiculars from A to $B'C'$, from B to $C'A'$, from C to $A'B'$ are concurrent at the centroid G of ABC. Now since ABC and $A'B'C'$ are orthologic, with G as one of the orthology centers, by Theorem 2, the perpendiculars from A' to BC, from B' to CA, from C' to AB are concurrent at the centroid G' of $A'B'C'$. Hence, the symmedian point K coincides with the centroid of its pedal triangle.

Conversely, let P a point with pedal triangle $A'B'C'$, and suppose P is the centroid of $A'B'C'$; it has homogeneous barycentric coordinates $(1 : 1 : 1)$ with reference to $A'B'C'$. Since ABC and $A'B'C'$ are orthologic, by Theorem 2, we have that the perpendiculars from A to $B'C'$, from B to $C'A'$, from C to $A'B'$ are concurrent at a point Q with homogeneous barycentric coordinates $(1 : 1 : 1)$ with reference to ABC. This is the centroid G. By Lemma 1, this is also the isogonal conjugate of P. This shows that $P = K$, the Lemoine (symmedian) point.

This completes the proof of Lemoine's theorem.

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