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A New Proof of a Weighted Erdős-Mordell Type Inequality

Yu-Dong Wu

Dedicated to Miss Xiao-Ping Lü
on the occasion of the 24-th Teachers' Day

Abstract. In this short note, by making use of one of Liu's theorems and Cauchy-Schwarz Inequality, we solve a conjecture posed by Liu [3] and give a new proof of a weighted Erdős–Mordell type inequality. Some interesting corollaries are also given at the end.

1. Introduction and Main Results

Let P be an arbitrary point in the plane of triangle ABC. Denote by R_1, R_2, and R_3 the distances from P to the vertices A, B, and C, and r_1, r_2, and r_3 the signed distances from P to the sidelines BC, CA, and AB, respectively. The neat and famous inequality

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3), \tag{1}$$

conjectured by Paul Erdős in 1935, was first proved by L. J. Mordell and D. F. Barrow (see [2]). In 2005, Jian Liu [3] obtained a weighted Erdős-Mordell type inequality as follows.

Theorem 1. *For* $x, y, z \in \mathbb{R}$,

$$\begin{aligned} &x^2\sqrt{R_2+R_3} + y^2\sqrt{R_3+R_1} + z^2\sqrt{R_1+R_2} \\ \geq &\sqrt{2}(yz\sqrt{r_2+r_3} + zx\sqrt{r_3+r_1} + xy\sqrt{r_1+r_2}). \end{aligned} \tag{2}$$

Liu's proof, however, is quite complicated. We give a simple proof of Theorem 1 as a corollary of a more general result, also conjectured by Liu in [3].

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Theorem 2. *For $x, y, z \in \mathbb{R}$ and arbitrary positive real numbers u, v, w, we have*

$$\begin{aligned}
&x^2\sqrt{v+w}+y^2\sqrt{w+u}+z^2\sqrt{u+v}\\
\geq&2\left(yz\sqrt{u\sin\frac{A}{2}}+zx\sqrt{v\sin\frac{B}{2}}+xy\sqrt{w\sin\frac{C}{2}}\right).
\end{aligned} \tag{3}$$

2. Preliminary Results

In order to prove our main results, we shall require the following two lemmas.

Lemma 3 ([4, 5])**.** *For $x, y, z \in \mathbb{R}$, $p_i \in (-\infty, 0)\bigcup(0, +\infty)$, and $q_i \in \mathbb{R}$ for $i = 1, 2, 3$, the quadratic inequality of three variables*

$$p_1x^2+p_2y^2+p_3z^2 \geq q_1yz+q_2zx+q_3xy$$

holds if and only if

$$\begin{cases}p_i>0,\ i=1,\ 2,\ 3;\\ 4p_2p_3>q_1^2,\ 4p_3p_1>q_2^2,\ 4p_1p_2>q_3^2,\\ 4p_1p_2p_3\geq p_1q_1^2+p_2q_2^2+p_3q_3^2+q_1q_2q_3.\end{cases}$$

Lemma 4. *In $\triangle ABC$, we have*

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=1.$$

Proof. This follows from the formula $\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$ and the known identity

$$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

□

3. Proof of Theorem 2

(1) For $u, v, w > 0$,

$$\begin{cases}\sqrt{v+w}>0,\\ \sqrt{w+u}>0,\\ \sqrt{u+v}>0.\end{cases} \tag{4}$$

(2) From $\sin\frac{A}{2}, \sin\frac{B}{2}, \sin\frac{C}{2} \in (0,1)$, we easily get

$$\begin{cases}4\sqrt{(w+u)(u+v)}>4u>4u\sin\frac{A}{2},\\ 4\sqrt{(u+v)(v+w)}>4v>4v\sin\frac{B}{2},\\ 4\sqrt{(v+w)(w+u)}>4w>4w\sin\frac{C}{2}.\end{cases} \tag{5}$$

By the Cauchy-Schwarz inequality and Lemma 4, we have

$$\begin{aligned}&\left(u\sqrt{v+w}\sin\frac{A}{2}+v\sqrt{w+u}\sin\frac{B}{2}+w\sqrt{u+v}\sin\frac{C}{2}+\sqrt{2uvw}\sqrt{2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\right)^2\\ \le&\left(u^2(v+w)+v^2(w+u)+w^2(u+v)+2uvw\right)\\ &\cdot\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\\ =&(u+v)(v+w)(w+u).\end{aligned} \tag{6}$$

From Lemma 3 and (4)–(6), we conclude that inequality (3) holds. The proof of Theorem 2 is complete.

4. Applications of Theorem 2

Proof of Theorem 1. If we take $u = R_1$, $v = R_2$, $w = R_3$ and with known inequalities (see [1])

$$2R_1\sin\frac{A}{2}\ge r_2+r_3,\quad 2R_2\sin\frac{B}{2}\ge r_3+r_1,\quad 2R_3\sin\frac{C}{2}\ge r_1+r_2,$$

we obtain Theorem 1 immediately. This completes the proof of Theorem 1.

Many further inequalities can be obtained from various substitutions for (u, v, w). Here are two examples.

Corollary 5. *For $\triangle ABC$ and real numbers x, y, z, we have*

$$\begin{aligned}&x^2\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}+y^2\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}+z^2\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}\\ \ge&2\left(yz\sin\frac{A}{2}+zx\sin\frac{B}{2}+xy\sin\frac{C}{2}\right).\end{aligned}$$

Corollary 6. *For $\triangle ABC$ and real numbers x, y, z, we have*

$$\begin{aligned}&x^2\sqrt{\csc\frac{B}{2}+\csc\frac{C}{2}}+y^2\sqrt{\csc\frac{C}{2}+\csc\frac{A}{2}}+z^2\sqrt{\csc\frac{A}{2}+\csc\frac{B}{2}}\\ \ge&2(yz+zx+xy).\end{aligned}$$

Further inequalities can also be obtained from substitutions of (x, y, z) by geometric elements of $\triangle ABC$. The reader is invited to experiment with the possibilities.

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Yu-Dong Wu: Department of Mathematics, Zhejiang Xinchang High School, Shaoxing, Zhejiang 312500, P. R. China

E-mail address: yudongwu@foxmail.com, 39387088@qq.com