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Another Compass-Only Construction of the Golden Section and of the Regular Pentagon

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Abstract. We present a compass-only construction of the point dividing a *given* segment in the golden ratio. As a corollary, we obtain a very simple construction of a regular pentagon inscribed in a *given* circle.

Various constructions of the golden section and of the regular pentagon have already appeared in this journal. In particular, in $[1, 2]$, Kurt Hofstetter offers very interesting compass-only constructions that require only a small number of circles. However, the constructed divided segment and pentagon come into sight as fortunate outcomes of the completed figures and are not subject to any prior constraint. As a result, these constructions do not adjust easily to the usual cases when the segment to be divided or the circumcircle of the pentagon are given at the start. The purpose of this note is to propose direct, simple compass-only constructions adapted to such situations.

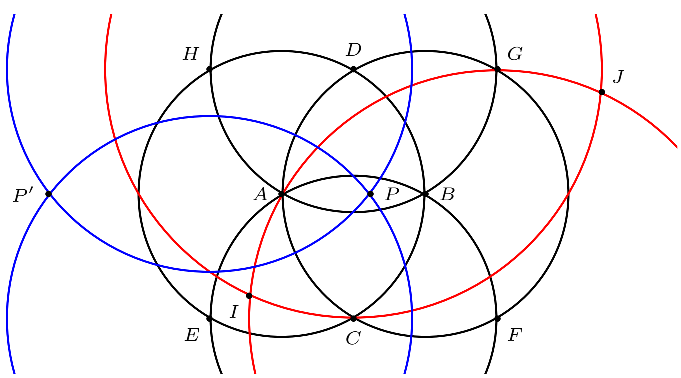

Figure 1

Construction 1. *Given two distinct points* A, B*, to obtain the point* P *of the line segment* AB *such that* $\frac{AP}{AB} = \frac{\sqrt{5}-1}{2}$*, construct*
(1) *with the same radius* AB*, the circles with centers* A *and* B*, to intersect at* C *and* D*,*
(2) *with the same radius* AB*, the circles with centers* C *and* D*, to intersect the two circles in* (1) *at* E, F, G, H (see Figure 1),

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(3) *with the same radius DC, the circles with centers D and F, to intersect at I and J,*
(4) *with the same radius BI, the circles with centers E and H.*
The points of intersection of these two circles are on the line AB, and P is the one between A and B.

Note that eight circles are needed, but if the line segment AB has been drawn, the number of circles drops to six, as it is easily checked. Note also that only three different radii are used.

Construction 2. *Given a point B on a circle Γ with center A, to obtain a regular pentagon inscribed in Γ with vertex B, construct*
(1) *the point P which divides AB in the golden section,*
(2) *the circle with center P and radius AB, to intersect Γ at B_1 and B_4,*
(3) *the circles $B_1(B)$ and $B_4(B)$ to intersect Γ, apart from B, at B_2 and B_3 respectively.*
The pentagon $BB_1B_2B_3B_4$ is the desired one (see Figure 2).

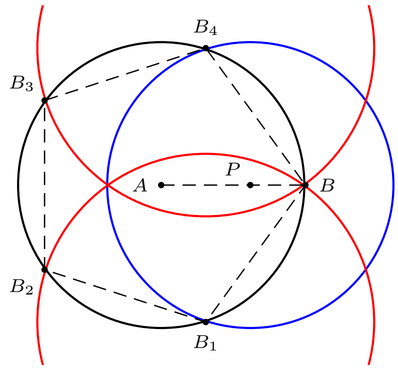

Figure 2

Proof of Construction 1. Let $a = AB$. Clearly, E, F (respectively F, D) are diametrically opposite on the circle with center C (respectively B) and radius a. It follows that EB is the perpendicular bisector of DF and since $IF = ID$, I is on the line EB. Therefore ΔIBF is right-angled at B, and $IB = a\sqrt{2}$ (since $IF = CD = a\sqrt{3}$ and $BF = a$). Now, the circles in (4) do intersect (since $HE = CD < 2BI$) and are symmetrical in the line AB, hence their intersections P, P' are certainly on this line. As for the relation $AP = \frac{\sqrt{5}-1}{2}AB$, it directly results from the following key property:

Let triangle BAE satisfy $AE = AB = a$ and $\angle BAE = 120°$ and let P be on the side AB such that $EP = a\sqrt{2}$. Then $AP = \frac{\sqrt{5}-1}{2}a$ (see Figure 3).

Indeed, the law of cosines yields $PE^2 = AE^2 + AP^2 - 2AE \cdot AP \cdot \cos 120°$ and this shows that AP is the positive solution to the quadratic $x^2 + ax - a^2 = 0$. Thus, $AP = \frac{\sqrt{5}-1}{2}a$. □

Note that $AP' = \frac{\sqrt{5}+1}{2}a$ is readily obtained in a similar manner.

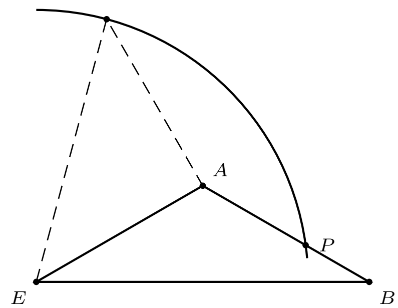

Figure 3

Proof of Construction 2. Since ΔAB_4P is isosceles with $B_4A = B_4P = a$, we have $\cos BAB_4 = \frac{1}{2}\frac{AP}{a} = \frac{\sqrt{5}-1}{4}$. Hence $\angle BAB_4 = 72°$ and the result immediately follows. □

As a final remark, Figure 3 and the property above lead to a quick construction of the golden section with ruler and compass.

References

[1] K. Hofstetter, A simple construction of the golden section, *Forum Geom.*, 2 (2002) 65–66.
[2] K. Hofstetter, A simple compass-only construction of the regular pentagon, *Forum Geom.*, 8 (2008) 147–148.

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