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Second-Degree Involutory Symbolic Substitutions

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Abstract. Suppose a, b, c are algebraic indeterminates. The mapping $(a, b, c) \to (bc, ca, ab)$ is an example of a second-degree involutory symbolic substitution (SISS) which maps the transfigured plane of a triangle to itself. The main result is a classification of SISSs as four individual mappings and two families of mappings. The SISS $(a, b, c) \to (bc, ca, ab)$ maps the circumcircle onto the Steiner ellipse. This and other examples are considered.

1. Introduction

This article is a sequel to [2], in which symbolic substitutions are introduced. A brief summary follows. The symbols a, b, c are algebraic indeterminates over the field of complex numbers. Suppose α, β, γ are nonzero homogeneous algebraic functions of (a, b, c) :

$$\alpha(a, b, c), \beta(a, b, c), \gamma(a, b, c), \tag{1}$$

all of the same degree of homogeneity. Throughout this work, triples with notations such as $U = (u, v, w)$ and $X = (x, y, z)$ are understood to be as in (1), except that one or two (but not all three) of the coordinates can be 0. Triples (x, y, z) and (x', y', z') are *equivalent* if $xy' = yx'$ and $yz' = zy'$. The equivalence class containing any particular (x, y, z) is denoted by $x : y : z$ and is a *point*. The set of points is the *transfigured plane*, denoted by $\mathcal{P}$. A well known model of $\mathcal{P}$ is obtained by taking a, b, c to be sidelengths of a euclidean triangle ABC and taking $x : y : z$ to be the point whose directed distances [1] from the sidelines BC, CA, AB are respectively proportional to x, y, z.

A simple example of a symbolic substitution is indicated by

$$(a, b, c) \to (bc, ca, ab).$$

This means that a point

$$x : y : z = x(a, b, c) : y(a, b, c) : z(a, b, c) \tag{2}$$

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[1]The coordinates $x : y : z$ are *homogeneous trilinear coordinates*, or simply *trilinears*. The notation (x, y, z), in this paper, represents an ordinary ordered triple, as when x, y, z are actual directed distances or when (x, y, z) is the argument of a function. Unfortunately, the notation (x, y, z) has sometimes been used for homogeneous coordinates, so that, for example $(2x, 2y, 2z) = (x, y, z)$, which departs from ordinary ordered triple notation. On the other hand, using colons, we have $2x : 2y : 2z = x : y : z$.

maps to the point

$$x : y : z = x(bc, ca, ab) : y(bc, ca, ab) : z(bc, ca, ab), \tag{3}$$

so that $\mathcal{P}$ is mapped to itself. We are interested in the effects of such substitutions on various points and curves. Consider, for example the Thompson cubic, $\mathcal{Z}(X_2, X_1)$, given by the equation[2]

$$bc\alpha(\beta^2 - \gamma^2) + ca\beta(\gamma^2 - \alpha^2) + ab\gamma(\alpha^2 - \beta^2) = 0. \tag{4}$$

For each point (2) on (4), the point (3) is on the cubic $\mathcal{Z}(X_6, X_1)$, given by the equation

$$a\alpha(\beta^2 - \gamma^2) + b\beta(\gamma^2 - \alpha^2) + cc(\alpha^2 - \beta^2) = 0. \tag{5}$$

Letting $\mathcal{S}(X_i)$ denote the image of X_i under the substitution $(a, b, c) \to (bc, ca, ab)$, specific points on $\mathcal{Z}(X_2, X_1)$ map to points on $\mathcal{Z}(X_6, X_1)$ as shown in Table 1:

Table 1. From $\mathcal{Z}(X_2, X_1)$to $\mathcal{Z}(X_6, X_1)$

X_i on $\mathcal{Z}(X_2, X_1)$	X_1	X_2	X_3	X_4	X_6	X_9	X_{57}
$\mathcal{S}(X_i)$ on $\mathcal{Z}(X_6, X_1)$	X_1	X_6	X_{194}	X_{3224}	X_2	X_{43}	X_{87}

As suggested by Table 1, $\mathcal{S}(\mathcal{S}(X)) = X$ for every X, which is to say that $\mathcal{S}$ is involutory. The main purpose of this article is to find explicitly all second-degree involutory symbolic substitutions.

2. Terminology and Examples

A *polynomial triangle center* is a point U which has a representation

$$u(a, b, c) : v(a, b, c) : w(a, b, c),$$

where $u(a, b, c)$ is a polynomial in a, b, c and these conditions hold:

$$v(a, b, c) = \quad u(b, c, a); \tag{6}$$

$$w(a, b, c) = \quad u(c, a, b); \tag{7}$$

$$|u(a, c, b)| = \quad |u(a, b, c)|\,. \tag{8}$$

If $u(a, b, c)$ has degree 2, then U is a *second-degree triangle center*. A *second-degree symbolic substitution* is a transformation of $\mathcal{P}$ or some subsert thereof, with images in $\mathcal{P}$, given by a symbolic substitution of the form

$$(a, b, c) \longrightarrow (u(a, b, c), v(a, b, c), w(a, b, c))$$

for some second-degree triangle center U. The mapping (whether of polynomial form or not) is *involutory* if its compositional square is the identity; that is, if

$$u(u, v, w) : v(u, v, w) : w(u, v, w) = a : b : c,$$

[2]Triangle centers are indexed as in [1]: $X_1 =$ incenter, $X_2 =$ centroid, etc. The cubic $\mathcal{Z}(U, P)$ is defined as the set of points $\alpha : \beta : \gamma$ satisfying

$$up\alpha(q\beta^2 - r\gamma^2) + vq\beta(r\gamma^2 - p\alpha^2) + wr\gamma(p\alpha^2 - q\beta^2) = 0$$

where $U = u : v : w$ and $P = p : q : r$. Geometrically, $\mathcal{Z}(U, P)$ is the locus of $X = x : y : z$ such that the P-isoconjugate of X is on the line UX. The P-isoconjugate of X (and the X-isoconjugate of P) is the point $qryz : rpzx : pqxy$.

where

$$u = u(a, b, c),\ v = v(a, b, c),\ w = w(a, b, c).$$

Equivalently, a symbolic substitution $(a, b, c) \longrightarrow (u, v, w)$ is involutory if

$$u(u, v, w) = ta$$

for some function t of (a, b, c) that is symmetric in a, b, c. Henceforth we shall abbreviate "second-degree involutory symbolic substitution" as SISS. Following are four examples.

Example 1. The SISS

$$(a, b, c) \longrightarrow (bc, ca, ab) \tag{9}$$

gives

$$\begin{aligned} u(u, v, w) &= u(bc, ca, ab) \\ &= (bc)(ca) \\ &= ta, \end{aligned}$$

where $t = abc$.

Example 2. The SISS

$$(a, b, c) \longrightarrow (a^2 - bc, b^2 - ca, c^2 - ab) \tag{10}$$

gives

$$\begin{aligned} u(u, v, w) &= u(a^2 - bc, b^2 - ca, c^2 - ab) \\ &= (a^2 - bc)^2 - (b^2 - ca)(c^2 - ab) \\ &= ta, \end{aligned}$$

where

$$t = (a + b + c)\,(a^2 + b^2 + c^2 - bc - ca - ab).$$

Note that (10) is meaningless for $a = b = c$. As a, b, c, are indeterminates, however, such cases do not require additional writing, just as, when one writes "$\tan\theta$" where θ is a variable, it is understood that $\theta \neq \frac{\pi}{2}$.

Example 3. The SISS

$$(a, b, c) \longrightarrow (b^2 + c^2 - ab - ac, c^2 + a^2 - bc - ba, a^2 + b^2 - ca - cb) \tag{11}$$

gives

$$u(u, v, w) = ta,$$

where

$$t = 2\,(a + b + c)\,(a^2 + b^2 + c^2 - bc - ca - ab).$$

Example 4. The SISS

$$(a, b, c) \longrightarrow (a(a - b - c), b(b - c - a), c(c - a - b)) \tag{12}$$

gives

$$u(u, v, w) = ta,$$

where

$$t = (a - b - c)\,(b - c - a)(c - a - b).$$

3. Main result

Theorem. *In addition to the four SISSs (9)-(12), there are two families of SISSs given below by (17) and (18), and there is no other SISS.*

Proof. Equations (6) –(8) and the requirement that u be a polynomial of degree 2 imply that u is expressible in one of these two forms:

$$u = ja^2 + k(b^2 + c^2) + lbc + ma(b + c) \tag{13A}$$

$$u = (b - c)(ja + k(b + c)) \tag{14}$$

for some complex numbers j, k, l, m. The proof will be given in two parts, depending on (13A) and (14).

Part 1: u given by (13A). In this case,

$$v = jb^2 + k(c^2 + a^2) + lca + mb(c + a), \tag{13B}$$

$$w = jc^2 + k(a^2 + b^2) + lab + mc(a + b). \tag{13C}$$

Let $P = u(u, v, w)$. We wish to find all j, k, l, m for which P factors as ta, where t is symmetric in a, b, c. The polynomial P can be written as $aQ + R$, where Q and R are polynomials and the R is invariant of a. In order to have $P = ta$, the coefficients j, k, l, m must make $R(a, b, c)$ identically 0. We have

$$R = (b^4 + c^4)S_1 + 2bc(b^2 + c^2)S_2 + b^2c^2S_3,$$

where

$$S_1 = jkl + jkm + k^3 + jk^2 + j^2k + k^2m,$$

$$S_2 = jkl + jkm + jlm + klm + km^2 + k^2m,$$

$$S_3 = 2jkm + 6jk^2 + jl^2 + j^2l + k^2l + 2km^2 + 2k^2m + 3lm^2.$$

Thus, we seek j, k, l, m for which $S_1 = S_2 = S_3 = 0$.

Case 1: $j = 0$. Here,
$S_1 = (k + m)\,k^2$, so that $k = 0$ or $k = -m$.
$S_2 = mk\,(k + l + m)$, so that $m = 0$ or $k = 0$ or $k + l + m = 0$.
$S_3 = k^2l + 2km^2 + 2k^2m + 3lm^2$.

Subcase 1.1: $j = 0$ and $k = 0$. Here, $S_2 = 0$, $S_3 = 3lm^2$, so that $l = 0$ or $m = 0$ but not both. If $l = 0$ and $m \neq 0$, then by (13A-C),

$$P = mu(v + w) = -m^3a\,(ab + ac + 2bc)\,(b + c)\,,$$

not of the required form aQ where Q is symmetric in a, b, c. On the other hand, if $m = 0$ and $l \neq 0$, then $P = lvw = l^3a^2bc$, so that, on putting $l = 1$, we have $(u, v, w) = (bc, ab, ca)$, as in (9).

Subcase 1.2: $j = 0$ and $k = -m \neq 0$. Here, with $S_2 = 0$, $k \neq 0$, $m \neq 0$, and $k + l + m = 0$, we have $l = 0$, and (13A-C) give

$$P = k(v^2 + w^2) - ku(v + w) = 2a\,(a + b + c)\left(a^2 + b^2 + c^2 - bc - ca - ab\right)k^3,$$

so that taking $(j, k, l, m) = (0, 1, 0, -1)$ gives the SISS (11).

Case 2: $k = 0$. Here, $S_1 = 0$, $S_2 = jlm$, and $S_3 = l\left(jl + j^2 + 3m^2\right)$.

Subcase 2.1: $k = 0$ and $j = 0$. Here, since $S_3 = 0$, we have $3lm^2 = 0$. If $l = 0$, then

$$\begin{aligned} u &= ma(b + c), v = mb(c + a), w = mc(a + b), \\ P &= mu(v + w) = -\,(ab + ac + 2bc)\,(b + c)\,am^3, \end{aligned}$$

not of the required form aQ. On the other hand, if $m = 0$, then

$$u = lbc, \quad v = lca, \quad w = lab,$$

so that taking $(j, k, l, m) = (0, 0, 1, 0)$ gives the SISS (9).

Subcase 2.2: $k = 0$ and $l = 0$. Here, $S_2 = S_3 = 0$, and (13A-C) give

$$\begin{aligned} P &= ju^2 + mu(v + w) \\ &= a\,(aj + bm + cm) \\ &\quad \cdot \left(abjm + acjm + abm^2 + acm^2 + 2bcm^2 + b^2jm + c^2jm + a^2j^2\right). \end{aligned}$$

In order for P to have the form aQ with Q symmetric in a, b, c, the factor

$$\left(abjm + acjm + abm^2 + acm^2 + 2bcm^2 + b^2jm + c^2jm + a^2j^2\right)$$

must factor as

$$(bj + cm + am)(cj + am + bm).$$

The identity

$$\begin{aligned} &\left(abjm + acjm + abm^2 + acm^2 + 2bcm^2 + b^2jm + c^2jm + a^2j^2\right) \\ &- (bj + cm + am)(cj + am + bm) \\ =\ &(m - j)\,(j + m)\left(bc - a^2\right) \end{aligned}$$

shows that this factorization occurs if and only if $j = m$ or $j = -m$. If $j = m$, then

$$u = a^2 + a(b + c), \quad v = b^2 + b(c + a), \quad w = c^2 + c(a + b),$$

leading to $(j, k, l, m) = (1, 0, 0, 1)$, but this is simply the identity substitution $(a, b, c) \to (a, b, c)$, not an SISS.

On the other hand, if $j = -m$, then

$$P = a\,(aj + bm + cm)\,(bj + cm + am)(cj + am + bm),$$

so that for $(j, k, l, m) = (1, 0, 0, -1)$, we have the SISS (12).

Subcase 2.3: $k=0$ and $m=0$. Here,

$$\begin{aligned}P &= ju^2+lvw\\ &= a^4j^3+a^2bcl^3+ab^3jl^2+ac^3jl^2+2a^2bcj^2l+b^2c^2jl^2+b^2c^2j^2l,\end{aligned}$$

which has the form aQ only if $b^2c^2jl^2+b^2c^2j^2l=0$, which means that $jl(j+l)=0$. If $j=0$ or $l=0$, we have solutions already found. If $j=-l$, then

$$\begin{aligned}P &= ju^2-jvw\\ &= l^3a\,(a+b+c)\left(ab+ac+bc-a^2-b^2-c^2\right),\end{aligned}$$

giving $(j,k,l,m)=(1,0,-1,0)$, the SISS (10).

Case 3: $l=0$. Here,

$$\begin{aligned}S_1 &= \left(jk+jm+km+j^2+k^2\right)k,\\ S_2 &= 2mk\,(j+k+m)\,,\\ S_3 &= 2k\left(3jk+jm+km+m^2\right).\end{aligned}$$

Subcase 3.1: $l=0$, $m=0$, $S_1=\left(jk+j^2+k^2\right)k$, $S_2=0$, and $S_3=6jk^2$. Since $S_3=0$, we have $j=0$ or $k=0$, already covered.

Subcase 3.2: $l=0$, and either $j=0$ or $k=0$, already covered.

Case 4: $m=0$. Here, $S_1=k\left(jk+jl+j^2+k^2\right)$, $S_2=2jkl$, and $S_3=\left(6jk^2+jl^2+j^2l+k^2l\right)$. Since $S_2=0$, we must have $j=0$ or $k=0$ or $l=0$. All of these possibilities are already covered.

Case 5: none of j,k,l,m is 0. Here,

$$\begin{aligned}S_1 &= k\left(jk+jl+jm+km+j^2+k^2\right),\\ S_2 &= jkl+jkm+jlm+klm+km^2+k^2m,\\ S_3 &= 2jkm+6jk^2+jl^2+j^2l+k^2l+2km^2+2k^2m+3lm^2.\end{aligned}$$

As $j\neq 0$ and $k\neq 0$, the requirement that $S_1=0$ gives

$$l=-\frac{jk+jm+km+j^2+k^2}{j}. \tag{15}$$

Substitute l into the expression for S_2 and factor, getting

$$S_2=-\frac{(k+m)(j+m)(jk+j^2+k^2)}{j}=0. \tag{16}$$

Subcase 5.1: $m=-j$. Here, $l=-\frac{k^2}{j}$. This implies $S_1=S_2=S_3=0$ and

$$P=\frac{a(ak+(a-b-c)j)(bk+(b-c-a)j)(ck+(c-a-b)j)(k-j)^3}{j^3},$$

which is of the form aQ with Q symmetric in a, b, c. Because of homogeneity, we can without loss of generality take $(j, k, l, m) = (1, k, -k^2, -1)$, where $k \notin \{0, 1, -2\}$. This leaves a family of SISSs:

$$(a, b, c) \to (u, v, w), \tag{17}$$

where

$$\begin{aligned} u &= a^2 + k(b^2 + c^2) - k^2 bc - a(b + c), \\ v &= b^2 + k(c^2 + a^2) - k^2 ca - b(c + a), \\ w &= c^2 + k(a^2 + b^2) - k^2 ab - c(a + b), \\ P &= a(k - 1)^3 (a - b - c + ak)(b - a - c + bk)(c - b - a + ck). \end{aligned}$$

Note that for $k = -2$, we have $u = (a + b + c)(a - 2b - 2c)$, so that the involutory substitution

$$(a, b, c) \to (a - 2b - 2c, b - 2c - 2a, c - 2a - 2b)$$

is actually of first-degree, not second. (It is easy to check that for

$$u = a + mb + mc,$$

the only values of m for which the substitution $(a, b, c) \to (u, v, w)$ is involutory are 0 and -2.)

Subcase 5.2: $m = -k$. Here, $l = -j$. This implies $S_1 = S_2 = S_3 = 0$ and

$$P = a(a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab)(j + 2k)(k - j)^2,$$

which is of the form aQ with Q symmetric in a, b, c. Thus, if $j \neq k$ and $j \neq -2k$, we take $(j, k, l, m) = (j, k, -j, -k)$ and have a family of SISSs:

$$(a, b, c) \to (u, v, w) \tag{18}$$

where

$$\begin{aligned} u &= a^2 j + b^2 k + c^2 k - bcj - abk - ack, \\ v &= b^2 j + c^2 k + a^2 k - caj - bck - bak, \\ w &= c^2 j + a^2 k + b^2 k - abj - cak - cbk. \end{aligned}$$

Note that $u = (a^2 - bc)j + (b^2 + c^2 - ab - ac)k$, a linear combination of second-degree polynomials appearing in (10) and (11).

Subcase 5.3: Equation (16) leaves one more subcase: $jk + j^2 + k^2 = 0$. This and (15) give $l = \frac{(j+k)k}{j}$, implying $S_1 = (k + m)(j + k)k$. Since $k \neq 0$ and $j + k \neq 0$ (because $l \neq 0$), we have $S_1 = 0$ only if $m = -k$, already covered in subcase 5.2.

Part 2: u given by (14). In this case,

$$P = (aj + bk + ck)(b - c)(j - k)\left(2bcj - acj - abj - 2a^2 k + b^2 k + c^2 k\right),$$

which is not, for any (j, k, l, m), of the form aQ where Q is symmetric in a, b, c. $\square$

4. Mappings by symbolic substitutions

To summarize from [2], a symbolic substitution $\mathcal{S}$ maps points to points, lines to lines, conics to conics, cubics to cubics, circumconics to circumconics, and inconics to inconics. Regarding cubics, $\mathcal{S}$ maps each cubic $\mathcal{Z}(U,P)$ to the cubic $\mathcal{Z}(\mathcal{S}(U),\mathcal{S}(P))$ and each cubic $\mathcal{ZP}(U,P)$ to the cubic $\mathcal{ZP}(\mathcal{S}(U),\mathcal{S}(P))$. Symbolic substitutions thus have in common with projections and collineations various incidence properties and degree-preserving properties. On the other hand, symbolic substitutions are fundamentally different from strictly geometric transformations: given an ordinary 2-dimensional triangle ABC and a point $X = x(a,b,c) : y(a,b,c) : z(a,b,c)$ there seems no opportunity to apply geometric methods for describing the image-point

$$\mathcal{S}(X) = x' : y' : z' = x(bc,ca,ab) : y(bc,ca,ab) : z(bc,ca,ab).$$

Algebraically, however, it is clear if X lies on the circumcircle, which is to say that X is on the locus $a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$, and if $\mathcal{S}$ is the symbolic substitution in (9), then $\mathcal{S}(X)$ satisfies $bcy'z' + caz'x' + abx'y' = 0$, which is to say that $\mathcal{S}(X)$ lies on the Steiner ellipse, $bc\beta\gamma + ca\gamma\alpha + ab\alpha\beta = 0$.

Table 2. From circumcircle Γ to Steiner ellipse $\mathbb{E}$

X_i on Γ	X_{98}	X_{99}	X_{100}	X_{101}	X_{105}	X_{106}	X_{110}	X_{111}
$\mathcal{S}(X_i)$ on $\mathbb{E}$	X_{3225}	X_{99}	X_{190}	X_{668}	X_{3226}	X_{3227}	X_{670}	X_{3228}

As a final example, note that the point $X_{101} = b-c : c-a : a-b$ is a fixed point of the SISS (10), as verified here:

$$b - c \to b^2 - ca - (c^2 - ab) = (a+b+c)(b-c).$$

Consequently, the line X_1X_6, given by the equation

$$(b-c)\alpha + (c-a)\beta + (a-b)\gamma = 0,$$

is left fixed by the SISS $\mathcal{S}$ in (10), as typified by Table 3.

Table 3. From X_1X_6 to X_1X_6

X_i on X_1X_6	X_1	X_6	X_9	X_{37}	X_{44}	X_{238}
$\mathcal{S}(X_i)$ on X_1X_6	X_1	X_{238}	X_{1757}	X_{518}	X_{44}	X_6

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