The Lost Daughters of Gergonne

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Abstract. Given a triangle center we can draw line segments from each vertex through the triangle center to the opposite side, this splits the triangle into six smaller triangles called daughters. Consider the following problem: Given a triangle S and a rule for finding a center find a triangle T, if possible, so that one of the daughters of T, when using the rule is S. We look at this problem for the incenter, median and Gergonne point.

1. Introduction

Joseph-Diaz Gergonne (1771–1859) was a famous French geometer who founded the *Annales de Gergonne*, the first purely mathematical journal. He served for a time in the army, was the chair of astronomy at the University of Montpellier, and to the best of our knowledge never misplaced a single daughter [3].

The "daughters" that we will be looking at come from triangle subdivision. Namely, for any well defined triangle center in the interior of the triangle one can draw line segments (or Cevians) connecting each vertex through the triangle center to the opposite edge. These line segments then subdivide the original triangle into six daughter triangles.

Given a triangle and a point it is easy to find the daughter triangles. We are interested in going the opposite direction.

Problem. Given a triangle S and a well defined rule for finding a triangle center; construct, if possible, a triangle T so that S is a daughter triangle of T for the given triangle center.

For instance suppose that we use the *incenter* as our triangle center (which can be found by taking the intersection of the angle bisectors). Then if we represent the angles of the triangle T by the triple (A, B, C) it easy to see that one daughter will have angles $\left(\frac{A}{2}, \frac{A}{2} + \frac{B}{2}, \frac{B}{2} + C\right)$, all the other daughter triangles are found by permuting A, B and C. Since this is a linear transformation this can be easily inverted. So if S has angles a, b and c then the possible candidates for T are $(2a, 2b - 2a, c - b + a)$, along with any permutation of a, b and c. It is easy to show that if the triangle S is not equilateral or an isoceles triangle with largest angle $\geq 90°$ then there is at least one non-degenerate T for S.

We could also use the *centroid* which for a triangle in the complex plane with vertices at 0, Z and W is $\frac{Z+W}{3}$. In particular if $[0, Z, W]$ is the location of the vertices of T then $\left[0, \frac{Z}{2}, \frac{Z+W}{3}\right]$ is the location of the vertices of a daughter of T. This map is easily inverted, let S be $[0, z, w]$ then we can choose T to be $[0, 2z, 3w - 2z]$. So, for example, if S is an equilateral triangle then we should choose T to be a triangle similar to one with side lengths 2, $\sqrt{7}$ and $\sqrt{13}$.

In this note we will be focusing on the case when our triangle center is the *Gergonne point*, which is found by the intersection of the line segments connecting the vertices of the triangle to the point of tangency of the incircle on the opposite edges (see Figures 2-5 for examples).

Unlike centroids where every triangle is a possible daughter, or incenters where all but $(60°, 60°, 60°)$, $(45°, 45°, 90°)$ and obtuse isoceles are daughters, there are many triangles which cannot be a Gergonne daughter. We call such triangles *the lost daughters of Gergonne*.

To see this pictorially if we again represent triangles as triples (A, B, C) of the angles, then each "oriented" triangle (up to similarity) is represented by a point in P, where P is the intersection of the plane $A + B + C = 180°$ with the positive orthant (see [1, 2, 5, 6] for previous applications of P). Note that P is an equilateral triangle where the points on the edges are degenerate triangles with an angle of $0°$ and the vertices are $(180°, 0°, 0°)$, $(0°, 180°, 0°)$, $(0°, 0°, 180°)$; the center of the triangle is $(60°, 60°, 60°)$. In Figure 1 we have plotted the location of the possible Gergonne daughters in P, the large white regions are the lost daughters.
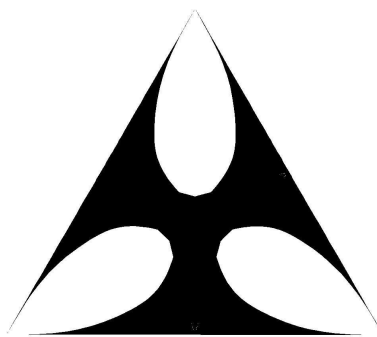

Figure 1. The possible Gergonne daughters in P.

2. Constructing T

We start by putting the triangle S into a standard position by putting one vertex at $(-1, 0)$ (with associated angle α), another vertex at $(0, 0)$ (with associated angle β) and the final vertex in the upper half plane. We now want to find (if possible) a triangle T which produces this Gergonne daughter in such a way that $(-1, 0)$ is a vertex and $(0, 0)$ is on an edge of T (see Figure 2). Since $(0, 0)$ will correspond to a point of tangency of the incircle we see that the incircle must be centered at $(0, t)$ with radius t for some positive t. Our method will be to solve for t in terms of α

and β. We will see that some values of α and β have no valid t, while others can have one or two.
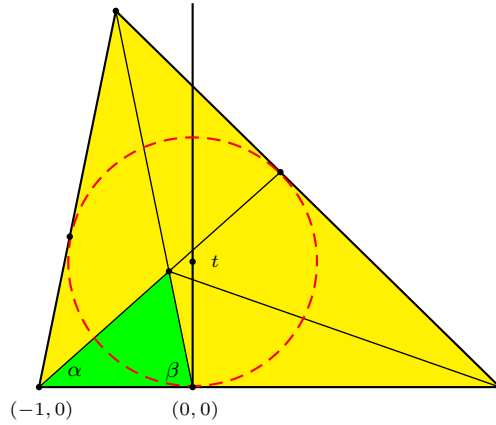

Figure 2.   A triangle in standard position.

Since the point of tangency of the incircle to the edge opposite $(-1, 0)$ must occur in the first quadrant, we immediately have that the angle α is acute and we will implicitly assume that in our calculations.

2.1. *The case $\beta = 90°$.* We begin by considering the special case $\beta = 90°$. In this setting it is easy to see that T must be an isosceles triangle of the form shown in Figure 3.
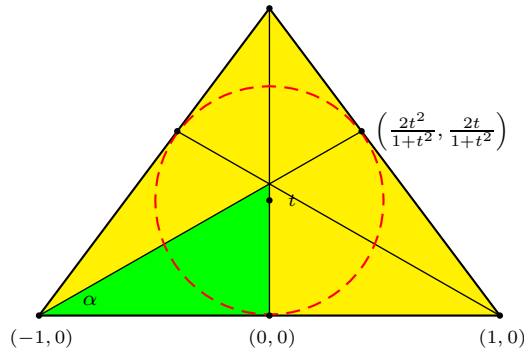

Figure 3.   The $\beta = 90°$ case.

The important part of Figure 3 is the location of the point $\left(\frac{2t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$. There are several ways to find this point. Ours will be to find the slope of the tangent line, then once this is found the point of tangency can easily be found. The key tool is the following lemma.

Lemma 1. *The slope m of the lines that pass through the point (p, q) and are tangent to the circle $x^2 + (y - t)^2 = t^2$ satisfy*

$$m^2 + \frac{2p(t - q)}{p^2 - t^2}m + \frac{q^2 - 2qt}{p^2 - t^2} = 0. \tag{1}$$

Proof. In order for the line $y = m(x - p) + q$ to be tangent to the circle $x^2 + (y - t)^2 = t^2$ the minimum distance between the line and $(0, t)$ must be t. Since the minimum distance between $(0, t)$ and the line $y = mx + (q - pm)$ is given by the formula $\frac{|t+pm-q|}{\sqrt{m^2+1}}$, we must have

$$t^2 = \left(\frac{|t + pm - q|}{\sqrt{m^2 + 1}} \right)^2.$$

Simplifying this relationship gives (1). □

Applying this with $(p, q) = (1, 0)$ we have that the slopes must satisfy,

$$m^2 + \frac{2t}{1 - t^2} m = 0.$$

We already know the solution $m = 0$, so the slope of the tangent line is $\frac{-2t}{1-t^2}$. Some simple algebra now gives us the point of tangency. We also have that the top vertex is located at $\left(0, \frac{2t}{1-t^2} \right)$.

Using the newly found point we must have

$$\tan \alpha = \frac{\frac{2t}{1+t^2}}{\frac{2t^2}{1+t^2} + 1} = \frac{2t}{1 + 3t^2},$$

which rearranges to

$$3(\tan \alpha)t^2 - 2t + \tan \alpha = 0, \text{ so that } t = \frac{1 \pm \sqrt{1 - 3\tan^2 \alpha}}{3 \tan \alpha}.$$

There are two restrictions. First, t must be real, and so we have $0 < \tan \alpha \le \frac{\sqrt{3}}{3}$, or $0 < \alpha \le 30°$. Second, $t < 1$ (if $t \ge 1$ then the triangle cannot close up), and so we need

$$\frac{1 + \sqrt{1 - 3\tan^2 \alpha}}{3 \tan \alpha} < 1 \text{ which reduces to } \tan \alpha > \frac{1}{2},$$

so for this root of t we need to have $\alpha > \arctan(1/2) \approx 26.565°$.

Theorem 2. *For $\beta = 90°$ and α given for a triangle S in standard position then*
(i) if $\alpha > 30°$ there is no T which produces S;
(ii) if $\alpha = 30°$ then the T which produces S is an equilateral triangle;
(iii) if $\arctan \frac{1}{2} < \alpha < 30°$ then there are two triangles T which produce S, these correspond to the two roots $t = \frac{1 \pm \sqrt{1-3\tan^2 \alpha}}{3 \tan \alpha}$;
(iv) if $\alpha \le \arctan \frac{1}{2}$ then there is one triangle T which produces S, this corresponds to the root $t = \frac{1 - \sqrt{1-3\tan^2 \alpha}}{3 \tan \alpha}$.

An example of the case when there can be two T is shown in Figure 4 for $\alpha = 29.85°$.
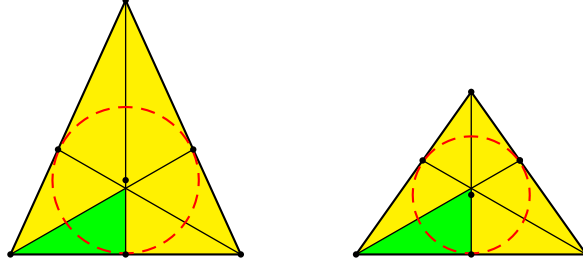
Figure 4. An example of a fixed S in standard position with two possible T.

2.2. *The case $\beta \neq 90°$.* Our approach is the same as in the previous case where we find the point of tangency opposite the vertex at $(-1,0)$ and then use a slope condition to restrict t. The only difference now is that finding the point takes a few more steps.

To start we can apply Lemma 1 with $(p,q) = (-1,0)$ and see that the slope of the line tangent to the circle is $\frac{2t}{1-t^2}$. The top vertex of T is then the intersection of the lines

$$y = \frac{2t}{1-t^2}(x+1) \text{ and } y = -(\tan\beta)x.$$

Solving for the point of intersection the top vertex is located at

$$(p^*, q^*) = \left(\frac{-2t}{2t + (1-t^2)\tan\beta}, \frac{2t\tan\beta}{2t + (1-t^2)\tan\beta} \right). \tag{2}$$

We can again apply Lemma 1 with (p^*, q^*) from (2), along with the fact that one of the two slopes is $\frac{2t}{1-t^2}$ to see that the slope of the edge opposite $(-1,0)$ is

$$m^* = \frac{2\tan\beta(t\tan\beta - 2)}{t^2\tan^2\beta - 4t\tan\beta + 4 - \tan^2\beta}.$$

It now is a simple matter to check that the point of tangency is

$$(x^*, y^*) = \left(\frac{(m^*)^2 p^* + tm^* - q^* m^*}{(m^*)^2 + 1}, \frac{(m^*)^2 t - p^* m^* + q^*}{(m^*)^2 + 1} \right).$$

We can also find that the x-intercept of the line, which will correspond to the final vertex of the triangle, is located at $\big(t\tan\beta/(t\tan\beta - 2), 0\big)$.

So as before we must have

$$\begin{aligned} \tan\alpha &= \frac{y^*}{x^* + 1} = \frac{(m^*)^2 t - p^* m^* + q^*}{(m^*)^2 p^* + tm^* - q^* m^* + (m^*)^2 + 1} \\ &= \frac{2t\tan^2\beta}{3t^2\tan^2\beta - 8t\tan\beta + \tan^2\beta + 4}. \end{aligned}$$

Which can be rearranged to give

$$(3\tan\alpha\tan^2\beta)t^2 - (2\tan^2\beta + 8\tan\alpha\tan\beta)t + (\tan\alpha\tan^2\beta + 4\tan\alpha) = 0.$$

Finally giving

$$t = \frac{\tan\beta + 4\tan\alpha \pm \sqrt{\tan^2\beta + 8\tan\alpha\tan\beta + 4\tan^2\alpha - 3\tan^2\alpha\tan^2\beta}}{3\tan\alpha\tan\beta}. \tag{3}$$

Theorem 3. *For $\beta \neq 90°$ and α given there are at most two triangles T which can produce S in standard position. These triangles T have vertices located at*

$$(-1,0), \left(\frac{-2t}{2t + (1 - t^2)\tan\beta}, \frac{2t\tan\beta}{2t + (1 - t^2)\tan\beta}\right), \text{ and } \left(\frac{t\tan\beta}{t\tan\beta - 2}, 0\right),$$

where t satisfies (3). *Further, we must have that t is positive and satisfies*

$$\frac{2}{\tan\beta} < t < \frac{1 + \sec\beta}{\tan\beta}.$$

Proof. The only thing left to prove are the bounds. For the upper bound, we must have that the second vertex is in the top half plane and so we need

$$\frac{2t\tan\beta}{2t + (1 - t^2)\tan\beta} > 0.$$

If $\tan\beta > 0$ then we need

$$2t + (1 - t^2)\tan\beta > 0 \text{ or } (\tan\beta)t^2 - 2t - \tan\beta < 0.$$

This is an upward facing parabola with negative y-intercept and so we need that t is less than the largest root, i.e.,

$$t < \frac{2 + \sqrt{4 + 4\tan^2\beta}}{2\tan\beta} = \frac{1 + \sec\beta}{\tan\beta}.$$

The case for $\tan\beta < 0$ is handled similarly.

For the lower bound we must have that the x-coordinate of the third vertex is positive. If $\tan\beta < 0$ this is trivially satisfied. If $\tan\beta > 0$ then we need $t\tan\beta - 2 > 0$ giving the bound. $\square$

As an example, if we let $\alpha = \beta = 45°$, then (3) gives $t = \frac{5 \pm \sqrt{10}}{3} \approx 0.6125$, or 2.7207. But neither of these satisfy $2 < t < 1 + \sqrt{2}$, so there is no T for this S. Combined with Theorem 2 this shows that $(45°, 45°, 90°)$ is a lost daughter of Gergonne.

On the other hand if we let $\alpha = \beta = 60°$ then (3) gives $t = \frac{\sqrt{3}}{3}, \frac{7\sqrt{3}}{9}$. The value $\frac{\sqrt{3}}{3}$ falls outside the range of allowable t, but the other one does fall in the range. The resulting triangle is shown in Figure 5 and has side lengths $\frac{19}{5}$, 8 and $\frac{49}{5}$.

3. Concluding comments

We now have a way given a triangle S to construct, if possible, a triangle T so that S is a Gergonne daughter of T. Using this it is possible to characterize triangles which are not Gergonne daughters. One can then look at what triangles are not Gergonne granddaughters (i.e., triangles which can be formed by repeating the subdivision rule on the daughters). Figure 6 shows the location of the Gergonne
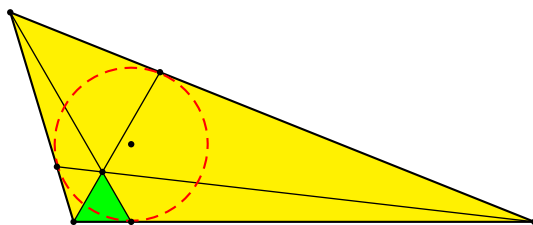
Figure 5. The unique triangle which has an equilateral triangle as a Gergonne daughter.

granddaughters in P. It can be shown the triangle in Figure 5 is not a Gergonne daughter, so that the equilateral triangle is not a Gergonne granddaughter.
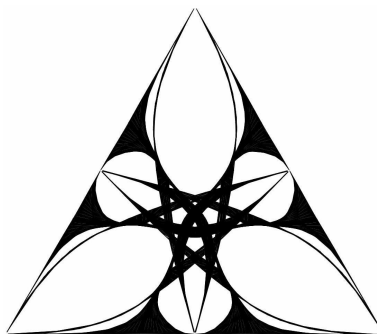

Figure 6. The possible Gergonne granddaughters in P.

One interesting problem is to find what triangles (up to similarity) can occur if we repeat the subdivision rule arbitrarily many times (see [2])? One example of this would be any triangle which is similar to one of its Gergonne daughters. Do any such triangles exist? (For the incenter there are only two such triangles, $(36°, 72°, 72°)$ and $(40°, 60°, 80°)$; for the centroid there is none.)

Besides the incenter, centroid and Gergonne point there are many other possible center points to consider (see the Encyclopedia of Triangle Centers [4] for a complete listing of well known center points, along with many others). One interesting point would be the Lemoine point, which can have up to *three* triangles T for a triangle S in standard position (as compared to 2 for the Gergonne point and 1 for the centroid).

References

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