Forum Geometricorum
Volume 9 (2009) 27–39.

FORUM GEOM
ISSN 1534-1178

Mappings Associated with Vertex Triangles

Clark Kimberling

Abstract. Methods of linear algebra are applied to triangle geometry. The vertex triangle of distinct circumcevian triangles is proved to be perspective to the reference triangle ABC, and similar results hold for three other classes of vertex triangles. Homogeneous coordinates of the perspectors define four mappings $\mathcal{M}_i$ on pairs of points (U, X). Many triangles homothetic to ABC are examined, and properties of the four mappings are presented. In particular, $\mathcal{M}_i(U, X) = \mathcal{M}_i(X, U)$ for $i = 1, 2, 3, 4$, and $\mathcal{M}_1(U, \mathcal{M}_1(U, X)) = X$; for this reason, $\mathcal{M}_1(U, X)$ is given the name *U-vertex conjugate of X*. In the introduction of this work, *point* is defined algebraically as a homogeneous function of three variables. Subsequent definitions and methods include symbolic substitutions, which are strictly algebraic rather than geometric, but which have far-reaching geometric implications.

1. Introduction

In [1], H. S. M. Coxeter proved a number of geometric results using methods of linear algebra and homogenous trilinear coordinates. However, the fundamental notions of triangle geometry, such as point and line in [1] are of the traditional geometric sort. In the present paper, we begin with an algebraic definition of point.

Suppose a, b, c are variables (or indeterminates) over the field of complex numbers and that x, y, z are homogeneous algebraic functions of (a, b, c) :

$$x = x(a, b, c), \quad y = y(a, b, c), \quad z = z(a, b, c),$$

all of the same degree of homogeneity and not all identically zero. Triples (x, y, z) and (x_1, y_1, z_1) are *equivalent* if $xy_1 = yx_1$ and $yz_1 = zy_1$. The equivalence class containing any particular (x, y, z) is denoted by $x : y : z$ and is a *point.* Let

$$A = 1 : 0 : 0, \quad B = 0 : 1 : 0, \quad C = 0 : 0 : 1.$$

These three points define the *reference triangle* ABC. The set of all points is the *transfigured plane,* as in [6]. If we assign to a, b, c numerical values which are the sidelengths of a euclidean triangle, then $x : y : z$ are homogeneous coordinates (e.g., trilinear or barycentric) as in traditional geometry, and points as defined just above are then points in the plane of a euclidean triangle ABC.

Possibly the earliest treatment of triangle-related points as functions rather than two-dimensional points appears in [3]; in [3]–[9], points-as-functions methods

Publication Date: February 23, 2009. Communicating Editor: Paul Yiu.

lead to problems whose meanings and solutions are nongeometric but which have geometric consequences. Perhaps the most striking are symbolic substitutions [6]–[8], the latter typified by substituting bc, ca, ab for a, b, c respectively. To see the nongeometric character of this substitution, one can easily find values of a, b, c that are sidelengths of a euclidean triangle but bc, ca, ab are not – and yet, this substitution and others have deep geometric consequences, as they preserve collinearity, tangency, and algebraic degree of loci. (In §6, the symbolic substitution $(a, b, c) \to (bc, ca, ab)$ is again considered.)

Having started with an algebraic definition of "point" as in [3], we now use it as a basis for defining other *algebraic* objects. A *line* is a set of points $x : y : z$ such that $lx + my + nz = 0$ for some point $l : m : n$; in particular, the line of two points $p : q : r$ and $u : v : w$ is given by

$$\begin{vmatrix} x & y & z \\ p & q & r \\ u & v & w \end{vmatrix} = 0.$$

A *triangle* is a set of three points. Harmonic conjugacy, isogonal conjugacy, and classes of curves are likewise defined by algebraic equations that are familiar in the literature of geometry (e.g. [1], [5], [10], [12], and many nineteenth-century works), where they occur as consequences of geometric foundations, not as definitions The same is true for other relationships, such as concurrence of lines, collinearity of points, perspectivity of triangles, similarity, and homothety.

So far in this discussion, coordinates have been general homogeneous. In traditional triangle geometry, two specific systems of homogeneous coordinates are common: barycentric and trilinear. In order to define special points and curves, we shall use their traditional trilinear representations. Listed here are a few examples: the centroid of ABC is *defined* as the point $1/a : 1/b : 1/c$; the line $\mathcal{L}^\infty$ at infinity, as $ax+by+cz = 0$. The isogonal conjugate of a point $x : y : z$ satisfying $xyz \neq 0$ is defined as the point $1/x : 1/y : 1/z$ and denoted by X^{-1}, and the circumcircle Γ is defined by $ayz + bzx + cxy = 0$, this being the set of isogonal conjugates of points on $\mathcal{L}^\infty$. Of course, we may illustrate definitions and relationships by evaluating a, b, c numerically—and then all the algebraic objects become geometric objects. (On the other hand, if, for example, $(a, b, c) = (6, 2, 3)$, then the algebraic objects remain intact even though there is no triangle with sidelengths $6, 2, 3$.)

Next, we define four classes of triangles. Suppose $X = x : y : z$ is a point not on a sideline of ABC; *i.e.*, $xyz \neq 0$. Let

$$A_1 = AX \cap BC = 0 : y : z$$
$$B_1 = BX \cap CA = x : 0 : z$$
$$C_1 = CX \cap AB = x : y : 0.$$

The triangle $A_1B_1C_1$ is the *cevian triangle* of X. Let A_2 be the point, other than A, in which the line AX meets Γ. Define B_2 and C_2 cyclically. The triangle $A_2B_2C_2$ is the *circumcevian triangle* of X, as indicated in Figure 1.

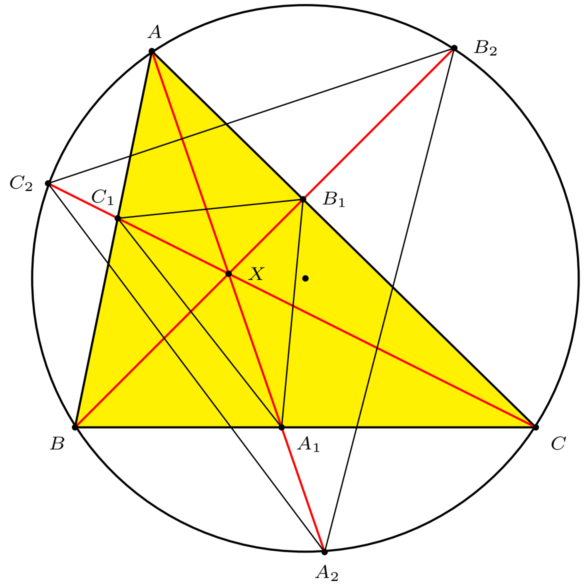

Figure 1.

Let A_3 be the $\{A, A_1\}$-harmonic conjugate of X (*i.e.*, $A_3 = -x : y : z$), and define B_3 and C_3 cyclically. Then $A_3B_3C_3$ is the *anticevian triangle* of X. Let

$$A' = BC \cap B_1C_1, \quad B' = CA \cap C_1A_1, \quad C' = AB \cap A_1B_1,$$

so that $A' = \{B, C\}$-harmonic conjugate of A_1 (*i.e.*, $A_1 = 0 : y : -z$), etc. The lines AA', BB', CC' are the *anticevians* of X, and the points

$$A_4 = AA' \cap \Gamma, \qquad B_4 = BB' \cap \Gamma, \qquad C_4 = CC' \cap \Gamma,$$

as in Figure 2, are the vertices of the *circum-anticevian triangle*, $A_4B_4C_4$, of X.

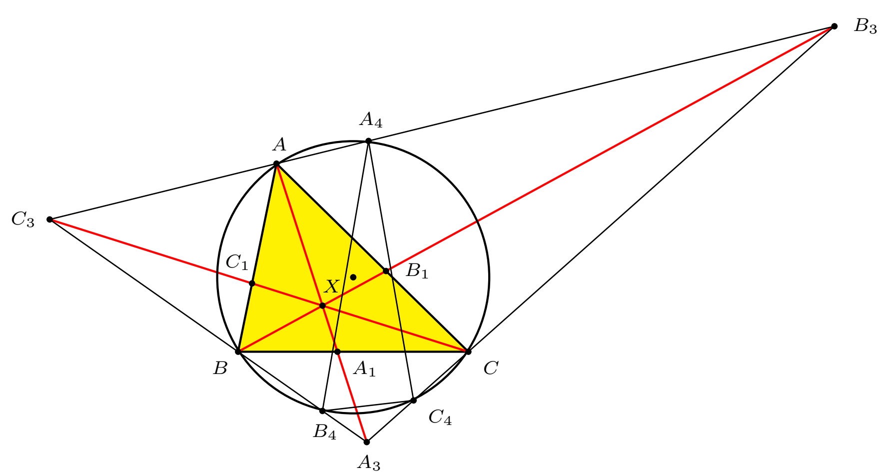

Figure 2.

With these four classes of triangles in mind, suppose $T = DEF$ and $T' = D'E'F'$ are triangles. The *vertex triangle* of T and T' is formed by the lines DD', EE', FF' as in Figure 3. Note that T and T' are perspective if and only if their vertex triangle is a single point.

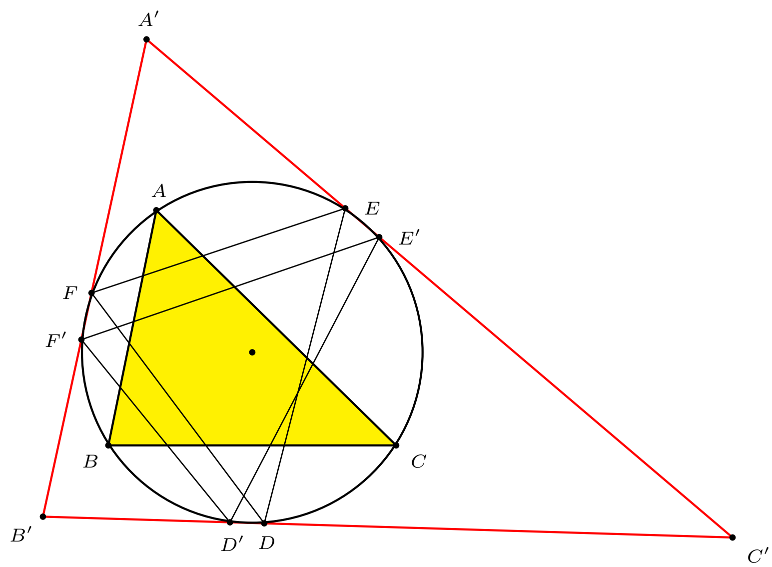

Figure 3.

2. The first mapping $\mathcal{M}_1$

Theorem 1. *The vertex triangle of distinct circumcevian triangles is perspective to ABC.*

Proof. Let $A'B'C'$ be the circumcevian triangle of $X = x : y : z$, and let $A''B''C''$ be the circumcevian triangle of $U = u : v : w$. The former can be represented as a matrix (e.g. [5, p.201]), as follows:

$$\begin{pmatrix} A' \\ B' \\ C' \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$$
$$= \begin{pmatrix} -ayz & (cy+bz)y & (bz+cy)z \\ (cx+az)x & -bzx & (az+cx)z \\ (bx+ay)x & (ay+bx)y & -cxy \end{pmatrix},$$

and likewise for $A''B''C''$ using vertices $u_i : v_i : w_i$ in place of $x_i : y_i : z_i$. Lines $A'A'', B'B'', C'C''$ are given by equations $x_i\alpha + y_i\beta + z_i\gamma = 0$ for $i = 4, 5, 6$, where

$$\begin{pmatrix} x_4 & y_4 & z_4 \\ x_5 & y_5 & z_5 \\ x_6 & y_6 & z_6 \end{pmatrix} = \begin{pmatrix} y_1w_1 - z_1v_1 & z_1u_1 - x_1w_1 & x_1v_1 - y_1u_1 \\ y_2w_2 - z_2v_2 & z_2u_2 - x_2w_2 & x_2v_2 - y_2u_2 \\ y_3w_3 - z_3v_3 & z_3u_3 - x_3w_3 & x_3v_3 - y_3u_3 \end{pmatrix},$$

so that the vertex triangle is given by

$$\begin{pmatrix} A''' \\ B''' \\ C''' \end{pmatrix} = \begin{pmatrix} x_7 & y_7 & z_7 \\ x_8 & y_8 & z_8 \\ x_9 & y_9 & z_9 \end{pmatrix} = \begin{pmatrix} y_5z_6 - z_5y_6 & z_5x_6 - x_5z_6 & x_5y_6 - y_5x_6 \\ y_6z_4 - z_6y_4 & z_6x_4 - x_6z_4 & x_6y_4 - y_6x_4 \\ y_4z_5 - z_4y_5 & z_4x_5 - x_4z_5 & x_4y_5 - y_4x_5 \end{pmatrix}. \tag{1}$$

The line AA''' thus has equation $0\alpha + z_7\beta - y_7\gamma = 0$, and equations for the lines BB''' and CC''' are obtained cyclically. The three lines concur if

$$\begin{vmatrix} 0 & z_7 & -y_7 \\ -z_8 & 0 & x_8 \\ y_9 & -x_9 & 0 \end{vmatrix} = 0,$$

and this is found (by computer) to be true. □

In connection with Theorem 1, the perspector is the point $P = x_8x_9 : x_8y_9 : z_8x_9$. After canceling long common factors, we obtain

$$\begin{aligned} P = {} & a/(a^2vwyz - ux(bw+cv)(bz+cy)) \\ & : b/(b^2wuzx - vy(cu+aw)(cx+az)) \\ & : c/(c^2uvxy - wz(av+bu)(ay+bx)). \end{aligned} \tag{2}$$

The right-hand side of (2) defines the first mapping, $\mathcal{M}_1(U, X)$. If U and X are triangle centers (defined algebraically, for example, in [3], [5], [11]), then so is $\mathcal{M}_1(U, X)$. It can be easily shown that $\mathcal{M}_1(U, X)$ is an involution; that is, $\mathcal{M}_1(\mathcal{M}_1(U, X)) = X$. In view of this property, we call $\mathcal{M}_1(U, X)$ the *U-vertex conjugate* of X. For example, the incenter-vertex conjugate of the circumcenter is the isogonal conjugate of the Bevan point; *i.e.*, $\mathcal{M}_1(X_1, X_3) = X_{84}$. The indexing of named triangle centers, such as X_{84}, is given in the *Encyclopedia of Triangle Centers* [9].

Vertex-conjugacy shares the *iso-* property with another kind of conjugacy called isoconjugacy; viz., the U-vertex-conjugate of X is the same as the X-vertex-conjugate of U. (The U-isoconjugate of X is the point $vwyz : wuzx : uvxy$; see the Glossary at [9].)

Other properties of $\mathcal{M}_1$ are given in §9 and in Gibert's work [2] on cubics associated with vertex conjugates.

3. The second mapping $\mathcal{M}_2$

Theorem 2. *The vertex triangle of distinct circum-anticevian triangles is perspective to ABC.*

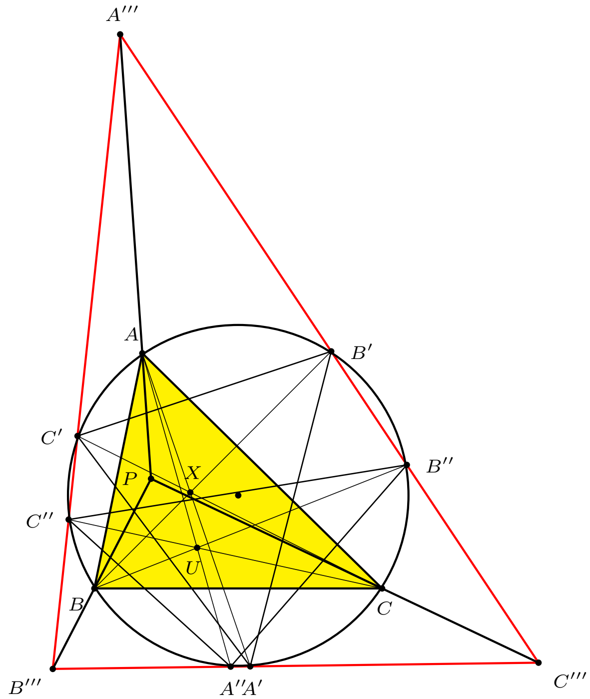

Figure 4.

Proof. The method of §2 applies, starting with

$$\begin{pmatrix} A' \\ B' \\ C' \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$$

$$= \begin{pmatrix} ayz & (cy-bz)y & (bz-cy)z \\ (cx-az)x & bzx & (az-cx)z \\ (bx-ay)x & (ay-bx)y & cxy \end{pmatrix},$$

and likewise for $A''B''C''$. □

The perspector is given by

$$P = p : q : r = \frac{a}{f(a,b,c,x,y,z)} : \frac{b}{f(b,c,a,y,z,a)} : \frac{c}{f(c,a,b,z,a,b))}, \quad (3)$$

where

$$f(a,b,c,s,y,z) = a^2vwyz - xu(bw-cv)(bz-cy),$$

and we define $\mathcal{M}_2(U,X) = P$ as in (3).

As this mapping is not involutory, we wish to solve the equation $P = \mathcal{M}_2(U,X)$ for X. The system to be solved, and the solution, are given by

$$\begin{pmatrix} g_1 & h_1 & k_1 \\ g_2 & h_2 & k_2 \\ g_3 & h_3 & k_3 \end{pmatrix} \begin{pmatrix} 1/x \\ 1/y \\ 1/z \end{pmatrix} = \begin{pmatrix} a/p \\ b/q \\ c/r \end{pmatrix}$$

and

$$\begin{pmatrix}1/x\\1/y\\1/z\end{pmatrix}=\begin{pmatrix}g_1&h_1&k_1\\g_2&h_2&k_2\\g_3&h_3&k_3\end{pmatrix}^{-1}\begin{pmatrix}a/p\\b/q\\c/r\end{pmatrix},$$

where

$$\begin{pmatrix}g_1&h_1&k_1\\g_2&h_2&k_2\\g_3&h_3&k_3\end{pmatrix}=\begin{pmatrix}a^2vw&-bu(bw-cv)&cu(bw-cv)\\av(cu-aw)&b^2wu&-cv(cu-aw)\\-aw(av-bu)&bw(av-bu)&k_3=c^2uv\end{pmatrix}.$$

Again, long factors cancel, leaving

$$X=x:y:z=g(a,b,c):g(b,c,a):g(c,a,b),$$

where

$$g(a,b,c)=\frac{a}{a^3qrv^2w^2-a^2G_2+aG_1+G_0},$$

where

$$\begin{aligned}G_0&=u^2p\,(bw-cv)^2\,(br+cq)\,,\\G_1&=uvwp\,(br-cq)\,(cv-bw)\,,\\G_2&=uvwrq\,(bw+cv)\,.\end{aligned}$$

4. The third mapping $\mathcal{M}_3$

Given a point $X=x:y:z$, we introduce a triangle $A'B'C'$ as follows:

$$\begin{pmatrix}A'\\B'\\C'\end{pmatrix}=\begin{pmatrix}x_1&y_1&z_1\\x_2&y_2&z_2\\x_3&y_3&z_3\end{pmatrix}=\begin{pmatrix}ayz&(cy+bz)y&(bz+cy)z\\(cx+az)x&bzx&(az+cx)z\\(bx+ay)x&(ay+bx)y&cxy\end{pmatrix},$$

and likewise for $A''B''C''$ in terms of $u:v:w$. The method of §2 shows that ABC is perspective to the vertex triangle of $A'B'C'$ and $A''B''C''$. The perspector is given by

$$\begin{aligned}\mathcal{M}_3(U,X)=\;&a/(a^2vwyz+xu(bw+cv)(bz+cy))\\&:b/(b^2wuzx+yv(cu+aw)(cx+az))\\&:c/(c^2uvxy+zw(av+bu)(ay+bx)).\end{aligned}\tag{4}$$

A formula for inversion is found as in §3: if $\mathcal{M}_3(U,X)=P=p:q:r$, then X is the point $g(a,b,c):g(b,c,a):g(c,a,b)$, where

$$g(a,b,c)=\frac{a}{a^3qrv^2w^2+a^2G_2-aG_1-G_0},$$

where

$$\begin{aligned}G_0&=u^2p\left(b^2w^2-c^2v^2\right)(br-cq)\,,\\G_1&=uvwp\,(br+cq)\,(bw+cv)\,,\\G_2&=uvwrq\,(bw+cv)\,.\end{aligned}$$

Geometrically, A' is the $\{A,\widehat{A}\}$-harmonic conjugate of $\widetilde{A}$, where $\widehat{A}\widehat{B}\widehat{C}$ and $\widetilde{A}\widetilde{B}\widetilde{C}$ are the cocevian and circumcevian triangles of X, respectively. (The vertices of the cocevian triangle of X are $\widehat{A} = 0 : z : -y$, $\widehat{B} = z : 0 : x$, $\widehat{C} = y : -x : 0$. The point $\widehat{A}$ is the $\{B, C\}$-harmonic conjugate of the A-vertex of the cevian triangle of U. The triangle $\widehat{A}\widehat{B}\widehat{C}$ is degenerate, as its vertices are collinear.)

5. The fourth mapping $\mathcal{M}_4$

For given $X = x : y : z$, define a triangle $A'B'C'$ by

$$\begin{pmatrix} A' \\ B' \\ C' \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = \begin{pmatrix} -ayz & (cy-bz)y & (bz-cy)z \\ (cx-az)x & -bzx & (az-cx)z \\ (bx-ay)x & (ay-bx)y & -cxy \end{pmatrix}.$$

Again, ABC is perspective to the vertex triangle of $A'B'C'$ and the triangle $A''B''C''$ similarly defined from U. The perspector is given by

$$\begin{aligned} \mathcal{M}_4(U, X) = & \; a/(a^2vwyz + xu(bw - cv)(bz - cy)) \\ & : b/(b^2wuzx + yv(cu - aw)(cx - az)) \\ & : c/(c^2uvxy + zw(av - bu)(ay - bx)). \end{aligned} \tag{5}$$

A formula for inversion is found as for §3: if $\mathcal{M}_4(U, X) = P = p : q : r$, then X is the point $g(a, b, c) : g(b, c, a) : g(c, a, b)$, where

$$g(a, b, c) = \frac{a}{(avw - buw - cuv)\,(a^2qrvw + up\,(cq - br)\,(bw - cv))}.$$

Geometrically, A' is the $\{A, \widehat{A}\}$-harmonic conjugate of $\widetilde{A}$, where $\widehat{A}\widehat{B}\widehat{C}$ and $\widetilde{A}\widetilde{B}\widetilde{C}$ are the cevian and circum-anticevian triangles of X, respectively.

6. Summary and extensions

To summarize §§2–5, vertex triangles associated with circumcevian and circum-anticevian triangles are perspective to the reference triangle ABC, and all four perspectors, given by (2-5), are representable by the following form for first trilinear coordinate:

$$\frac{a}{\frac{a^2}{ux} \pm \left(\frac{b}{v} \pm \frac{c}{w}\right)\left(\frac{b}{y} \pm \frac{c}{z}\right)}; \tag{6}$$

here, the three $\pm$ signs are limited to $-++$, $---$, $+++$, and $+--$, which correspond in order to the four mappings $\mathcal{M}_i(U, X)$.

Regarding each perspector $P = \mathcal{M}_i(U, X)$, formulas for the inverse mapping of X, for given U, have been given, and in the case of the first mapping, the transformation is involutory. The representation (6) shows that $\mathcal{M}_i(U, X) = \mathcal{M}_i(X, U)$ for each i, which is to say that $\mathcal{M}_i(U, X)$ can be viewed as a commutative binary operation. There are many interesting examples regarding the four mappings; some of them are given in §9.

For all four configurations, define $\mathcal{M}_i(U, U)$ by putting $x : y : z = u : v : w$ in (2)–(5), and note that (6) gives the perspector in these cases. In Figure 3, taking $X = U$ corresponds to moving E', F', G' to E, F, G so that in the limit the lines $B'C', C'A', A'B'$ are tangent to Γ. It would be of interest to know the set of triangle centers X for which there exists a triangle center U such that $\mathcal{M}_i(U, U) = X$.

The symbolic substitution

$$(a, b, c) \to (bc, ca, ab) \tag{7}$$

transforms the transfigured plane onto itself, as (7) maps each point $X = x : y : z = x(a, b, c) : y(a, b, c) : z(a, b, c)$ to the point

$$X' = x' : y' : z' = x(bc, ca, ab) : y(bc, ca, ab) : z(ca, ab, bc).$$

The circumcircle, as the locus of X satisfying $ayz + bzx + cxy = 0$, maps to the Steiner circumellipse [10], which is the locus of $x' : y' : z'$ satisfying

$$bcy'z' + caz'x' + abx'y' = 0.$$

Circumcevian triangles map to perspective triangles as in Theorem 1, and perspectors are given by applying (7) to (2). The substitution (7) likewise applies to the developments in §§3–5. Analogous (geometric) results spring from other (non-geometric) symbolic substitutions, such as $(a, b, c) \to (b + c, c + a, a + b)$ and $(a, b, c) \to (a^2, b^2, c^2)$.

7. Homothetic triangles

We return now to the vertex-triangles introduced in §§2–5 and establish that if U and X are a pair of isogonal conjugates, then their vertex triangle is homothetic to ABC.

Theorem 3. *Suppose X is a point not on a sideline of triangle ABC. Then the vertex triangle of the circumcevian triangles of X and X^{-1} is homothetic to ABC, and likewise for the pairs of vertex triangles in §§3–5.*

Proof. In accord with the definition of isogonal conjugate, trilinears for $U = X^{-1}$ are given by $u = yz$, $v = zx$, $w = xy$, so that

$$u : v : w = x^{-1} : y^{-1} : z^{-1}.$$

In the notation of §2, the vertex triangle (1) is given by its A-vertex $x_7 : y_7 : z_7$, where

$$\begin{aligned} x_7 =\ & abc(x^2 + y^2)(x^2 + z^2) + (a^2(bz + cy) + bc(by + cz))x^3 \\ & + a(a^2 + b^2 + c^2)x^2yz + (bcyz(bz + cy) + a^2yz(by + cz))x, \\ y_7 =\ & -bxz(ab(x^2 + y^2) + (a^2 + b^2 - c^2)xy), \\ z_7 =\ & -cxy(ac(x^2 + z^2) + (a^2 - b^2 + c^2)xz). \end{aligned}$$

Writing out the coordinates $x_8 : y_8 : z_8$ and $x_9 : y_9 : z_9$, we then evaluate the determinant for parallelism of sideline BC and the A-side of the vertex triangle:

$$\begin{vmatrix} a & b & c \\ 1 & 0 & 0 \\ y_8z_9 - z_8y_9 & z_8x_9 - x_8z_9 & x_8y_9 - y_8x_9 \end{vmatrix} = 0.$$

Likewise, the other two pairs of sides are parallel, so that the vertex triangle of the two circumcevian triangles is now proved homothetic to ABC.

The same procedure shows that the vertex triangles in §§3–5, when $U = X^{-1}$, are all homothetic to ABC (hence homothetic to one another, as well as the medial triangle, the anticomplementary triangle, and the Euler triangle). □

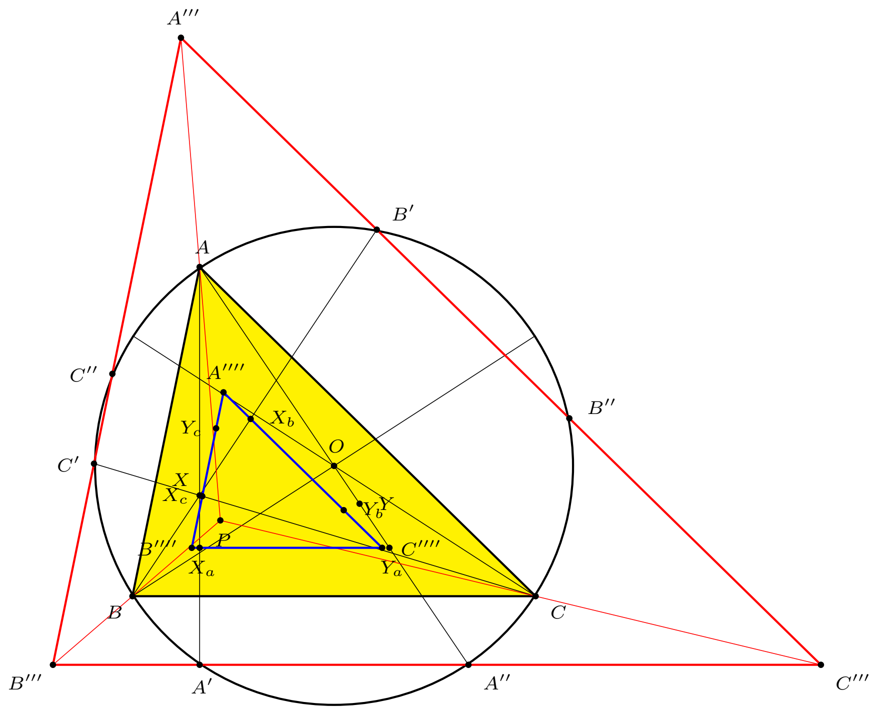

Figure 5.

Substituting into (6) gives a compact expression for the four classes of homothetic centers (*i.e.*, perspectors), given by the following first trilinear:

$$\frac{ayz}{(a^2 \pm (b^2 + c^2))yz \pm bc(y^2 + z^2)},$$

from which it is clear that the homothetic centers for X and X^{-1} are identical.

8. More Homotheties

Let $\mathcal{C}(X)$ denote the circumcevian triangle of a point X, and let O denote the circumcenter, as in Figure 6.

Theorem 4. *Suppose U is a point not on a sideline of triangle ABC. The vertex triangle of $\mathcal{C}(U)$ and $\mathcal{C}(O)$ is homothetic to the pedal triangle of U^{-1}.*

Proof. The vertex triangle $A'''B'''C'''$ of $C(U)$ and $C(O)$ is given by (1), using $U = u : v : w$ and

$$x = a(b^2 + c^2 - a^2), \quad y = b(c^2 + a^2 - b^2), \quad z = c(a^2 + b^2 - c^2).$$

Trilinears for A''' as initially computed include many factors. Canceling those common to all three trilinears leaves

$$x_7 = 4abc(avw + bwu + cuv) + u^2(a + b + c)(-a + b + c)(a - b + c)(a + b - c),$$
$$y_7 = uv(a^2 - b^2 + c^2)(b^2 - a^2 + c^2) - 2cw(a^2 + b^2 - c^2)(av + bu),$$
$$z_7 = uw(a^2 - c^2 + b^2)(c^2 - a^2 + b^2) - 2bv(a^2 + c^2 - b^2)(aw + cu),$$

and x_8, y_8, z_8 and x_9, y_9, z_9 are obtained from x_7, y_7, z_7 by cyclic permutations of a, b, c and u, v, w.

Since $U^{-1} = vw : wu : uv$, the vertices of the pedal triangle of U^{-1} are given ([5, p.186]) by

$$\begin{pmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{pmatrix} = \begin{pmatrix} 0 & w(u + vc_1) & v(u + wb_1) \\ w(v + uc_1) & 0 & u(v + wa_1) \\ v(w + ub_1) & u(w + va_1) & 0 \end{pmatrix},$$

where

$$(a_1, b_1, c_1) = (a(b^2 + c^2 - a^2), b(c^2 + a^2 - b^2), c(a^2 + b^2 - c^2)).$$

Side $B'''C'''$ of the vertex triangle is parallel to the corresponding sideline of the pedal triangle if the determinant

$$\begin{vmatrix} a & b & c \\ g_2h_3 - h_2g_3 & h_2f_3 - f_2h_3 & f_2g_3 - g_2f_3 \\ y_8z_9 - z_8y_9 & z_8x_9 - x_8z_9 & x_8y_9 - y_8x_9 \end{vmatrix} \tag{8}$$

equals 0. It is helpful to factor the polynomials in row 3 and cancel common factors. That and putting $f_1 = g_2 = h_3 = 0$ lead to the following determinant which is a factor of (8):

$$\begin{vmatrix} a & b & c \\ -h_2g_3 & h_2f_3 & f_2g_3 \\ 2a(bw + cv) & w(a^2 + b^2 - c^2) & v(a^2 - b^2 + c^2) \end{vmatrix}.$$

This determinant indeed equals 0. The parallelism of the other matching pairs of sides now follows cyclically. □

9. Properties of the four mappings

This section consists of properties of the mappings $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, $\mathcal{M}_4$ introduced in §§2–5. Proofs are readily given by use of well known formulas. In several cases, a computer is needed because of very lengthy trilinears. Throughout, it is assumed that neither U nor X lies on a sideline of ABC.

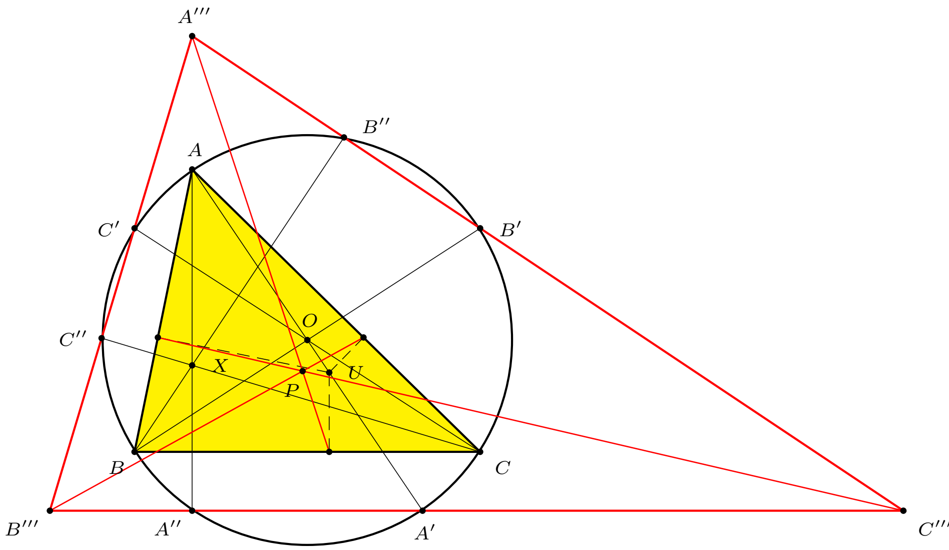

Figure 6.

1a. If $U \in \Gamma$, then $\mathcal{M}_1(U, X) = U$.

1b. If $X \notin \Gamma$, then

$$\mathcal{M}_1(X, X) = \frac{a}{ayz - bzx - cxy} : \frac{b}{bzx - cxy - ayz} : \frac{c}{cxy - ayz - bzx}.$$

If $U = \mathcal{M}_1(X, X)$, then

$$X = \frac{avw}{bw + cv} : \frac{bwu}{cu + aw} : \frac{cuv}{av + bu}.$$

1c. If X is the 1st Saragossa point of U, then $\mathcal{M}_1(U, X) = X$. (The 1st Saragossa point is the point

$$\frac{a}{bzx + cxy} : \frac{b}{cxy + ayz} : \frac{c}{ayz + bzx},$$

discussed at [9] just before X_{1166}.

1d. Suppose U is on the line at infinity, and let U^* be the isogonal conjugate of the antipode of the isogonal conjugate of U. Let L be the line X_3U^*. Then $\mathcal{M}_1(U, U^*) = X_3$, and if $X \in L$, then $\mathcal{M}_1(U, X)$ is the inverse-in-Γ of X.

1e. $\mathcal{M}_1$ maps the Darboux cubic to itself. (See [2] for a discussion of cubics associated with $\mathcal{M}_1$.

2a. $\mathcal{M}_2(X_6, X) = X$.

2b. $\mathcal{M}_2(X, X) = X$-Ceva conjugate of X_6.

2c. Let L be the line UX_6 and let L' be the line UU^c, where $U^c = \mathcal{M}_2(U, U)$. If $X \in L$, then $\mathcal{M}_2(U, X) \in L'$.

3a. $\mathcal{M}_3(X_6, X) = X$.

3b. If $X \in \Gamma$ and X is not on a sideline of ABC, then $\mathcal{M}_3(X, X)$ is the cevapoint X and X_6. (The cevapoint [9, Glossary] of points $P = p : q : r$ and $U = u : v : w$ is defined by trilinears

$$(pv + qu)(pw + ru) : (qw + rv)(qu + pv) : (ru + pw)(rv + qw).)$$

3c. If $U \in \Gamma$, then

$$\mathcal{M}_3(U, X) = \frac{u}{ayz - bzx - cxy} : \frac{v}{bzx - cxy - ayz} : \frac{w}{cxy - ayz - bzx},$$

which is the trilinear product $U \cdot \widehat{X}$, where $\widehat{X}$ is the X_2-isoconjugate of the X-Ceva conjugate of X_6.

4a. $\mathcal{M}_4(X_6, X) = X$.

4b. Suppose P is on the line at infinity (so that P^{-1} is on Γ). Let X be the cevapoint of X_6 and P. Then $\mathcal{M}_4(X_{251}, X) = P^{-1}$.

4c. Let $X^* = X \cdot \widehat{X}$, where $\widehat{X}$ is as in 3c. Then $\mathcal{M}_4(X, X^*) = X_6$.

References

[1] H. S. M. Coxeter, Some applications of trilinear coordinates, *Linear Algebra Appl.*, 226–228 (1995) 375–388.
[2] B. Gibert, Vertex conjugate cubics, available at http://pagesperso-orange.fr/bernard.gibert/Classes/c1050.html.
[3] C. Kimberling, Triangle centers as functions, *Rocky Mountain J. Math.*, 23 (1993) 1269–1286.
[4] C. Kimberling, Major centers of triangles, *Amer. Math. Monthly*, 104 (1997) 431–488.
[5] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.
[6] C. Kimberling, Symbolic substitutions in the transfigured plane of a triangle, *Aequationes Mathematicae*, 73 (2007) 156–171.
[7] C. Kimberling, Second-degree involutory symbolic substitutions, *Forum Geom.*, 8 (2008) 175–182.
[8] C. Kimberling, Cubics defined from symmetric functions, to appear in *Aequationes Mathematicae*.
[9] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.
[10] S. L. Loney, *The Elements of Coordinate Geometry*, Part II, Macmillan, London, 1957.
[11] E. Weisstein, *MathWorld*, http://mathworld.wolfram.com.
[12] P. Yiu, A tour of triangle geometry, http://www.math.fau.edu/yiu/Geometry.html.

Clark Kimberling: Department of Mathematics, University of Evansville, 1800 Lincoln Avenue, Evansville, Indiana 47722, U.S.A.

E-mail address: ck6@evansville.edu