Forum Geometricorum
Volume 9 (2009) 41–46.

FORUM GEOM
ISSN 1534-1178

On Integer Relations Between the Area and Perimeter of Heron Triangles

Allan J. MacLeod

Abstract. We discuss the relationship $P^2 = nA$ for a triangle with integer sides, with perimeter P and area A, where n is an integer. We show that the problem reduces to finding rational points of infinite order in a family of elliptic curves. The geometry of the curves plays a crucial role in finding real triangles.

1. Introduction

In a recent paper, Markov [2] discusses the problem of solving $A = mP$, where A is the area and P is the perimeter of an integer-sided triangle, and m is an integer. This relation forces A to be integral and so the triangle is always a Heron triangle.

In many ways, this is not a proper question to ask, since this relation is not scale-invariant. Doubling the sides to a similar triangle changes the area/perimeter ratio by a factor of 2. Basically, we have unbalanced dimensions - area is measured in square-units, perimeter in units but m is a dimensionless quantity.

It would seem much better to look for relations between A and P^2, which is the purpose of this paper. Another argument in favour of this is that the recent paper of Baloglou and Helfgott [1], on perimeters and areas, has the main equations (1) to (8) all balanced in terms of units.

We assume the triangle has sides (a, b, c) with $P = a + b + c$ and $s = \frac{P}{2}$. Then the area is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{4}\sqrt{P(P-2a)(P-2b)(P-2c)}$$

so that it is easy to see that $A < P^2/4$. Thus, to look for an integer link, we should study $P^2 = nA$ with $n > 4$.

It is easy to show that this bound on n can be increased quite significantly. We have

$$\frac{P^4}{A^2} = 16\frac{(a+b+c)^3}{(a+b-c)(a+c-b)(b+c-a)} \tag{1}$$

and we can, without loss of generality, assume $a = 1$. Then the ratio in equation (1) is minimised when $b = 1, c = 1$. This is obvious from symmetry, but can be easily proven by finding derivatives. Thus $\frac{P^4}{A^2} \geq 432$ and so $P^2 \geq 12\sqrt{3}A$ for all triangles, so we need only consider $n \geq 21$.

Publication Date: March 9, 2009. Communicating Editor: Paul Yiu.

As an early example of a solution, the $(3, 4, 5)$ triangle has $P^2 = 144$ and $A = 6$ so $n = 24$.

To proceed, we consider the equation

$$16\frac{(a+b+c)^3}{(a+b-c)(a+c-b)(b+c-a)} = n^2. \tag{2}$$

2. Elliptic Curve Formulation

Firstly, it is clear that we can let a, b, c be rational numbers, since a rational-sided solution is easily scaled up to an integer one.

From equation (2), we have

$$(n^2+16)a^3 + (48-n^2)(b+c)a^2 - (b^2(n^2-48) - 2bc(n^2+48) + c^2(n^2-48))a$$
$$+ (b+c)(b^2(n^2+16) + 2bc(16-n^2) + c^2(n^2+16)) = 0.$$

This cubic is very difficult to deal with directly, but a considerable simplification occurs if we use $c = P - a - b$, giving

$$4n^2(P-2b)a^2 - 4n^2(2b^2-3bP+P^2)a + P(4b^2n^2 - 4bn^2P + P^2(n^2+16)) = 0. \tag{3}$$

For this quadratic to have rational roots, we must have the discriminant being a rational square. This means that there must be rational solutions of

$$d^2 = 4n^2b^4 - 4n^2Pb^3 + n^2P^2b^2 + 32P^3b - 16P^4$$

and, if we define $y = \frac{2nd}{P^2}$ and $x = \frac{2nb}{P}$, we have

$$y^2 = x^4 - 2nx^3 + n^2x^2 + 64nx - 64n^2. \tag{4}$$

A quartic in this form is birationally equivalent to an elliptic curve, see Mordell [3]. Using standard transformations and some algebraic manipulation, we find the equivalent curves are

$$E_n : v^2 = u^3 + n^2u^2 + 128n^2u + 4096n^2 = u^3 + n^2(u+64)^2 \tag{5}$$

with the backward transformation

$$\frac{b}{P} = \frac{n(u-64)+v}{4nu}. \tag{6}$$

Thus, from a suitable point (u, v) on E_n, we can find b and P from this relation. To find a and c, we use the quadratic for a, but written as

$$a^2 - (P-b)a + \frac{P(16P^2 + n^2(P-2b)^2)}{4n^2(P-2b)} = 0. \tag{7}$$

The sum of the roots of this quadratic is $P - b = a + c$, so the two roots give a and c.

But, we should be very careful to note that the analysis based on equation (2) is just about relations between numbers, which could be negative. Even if they are all positive, they may not form a real-life triangle - they do not satisfy the triangle inequalities. Thus we need extra conditions to give solutions, namely $0 < a, b, c < \frac{P}{2}$.

3. Properties of E_n

The curves E_n are clearly symmetric about the u-axis. If the right-hand-side cubic has 1 real root R, then the curve has a single infinite component for $u \geq R$. If, however, there are 3 real roots $R_1 < R_2 < R_3$, then E_n consists of an infinite component for $u \geq R_3$ and a closed component for $R_1 \leq u \leq R_2$, usually called the "egg".

Investigating with the standard formulae for cubic roots, we find 3 real roots if $n^2 > 432$ and 1 real root if $n^2 < 432$. Since we assume $n \geq 21$, there must be 3 real roots and so 2 components. Descartes' rule of signs shows that all roots are negative.

It is clear that $u = -64$ does not give a point on the curve, but $u = -172$ gives $v^2 = 16(729n^2 - 318028)$ which is positive if $N \geq 21$. Thus we have $R_1 < -172 < R_2 < -64 < R_3 < 0$.

The theory of rational points on elliptic curves is an enormously developed one. The rational points form a finitely-generated Abelian group with the addition operation being the standard secant/tangent method. This group of points is isomorphic to the group $T \oplus \mathbb{Z}^r$, where T is one of $\mathbb{Z}_m,\ m = 1, 2, \ldots, 10, 12$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_m, m = 1, 2, 3, 4$, and r is the rank of the curve. T is known as the torsion-subgroup and consists of those points of finite order on the curve, including the point-at-infinity which is the identity of the group. Note that the form of E_n ensures that torsion points have integer coordinates by the Nagell-Lutz theorem, see Silverman and Tate [5].

We easily see the two points $(0, \pm 64n)$ and since they are points of inflexion of the curve, they have order 3. Thus T is one of $\mathbb{Z}_3, \mathbb{Z}_6, \mathbb{Z}_9, \mathbb{Z}_{12}, \mathbb{Z}_2 \oplus \mathbb{Z}_3$. Some of these possibilities would require a point of order 2 which correspond to integer zeroes of the cubic. Numerical investigations show that only $n = 27$, for $n \leq 499$, has an integer zero (at $u = -576$). Further investigations show $\mathbb{Z}_3$ as being the only torsion subgroup to appear, for $n \leq 499$, apart from $\mathbb{Z}_6$ for $n = -27$. Thus we conjecture that, apart from $n = 27$, the group of rational points is isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}^r$. The points of order 3 give $\frac{b}{P}$ undefined so we would need $r \geq 1$ to possibly have triangle solutions.

For $n = 27$, we find the point $H = (-144, 1296)$ of order 6, which gives the isosceles triangle $(5, 5, 8)$ with $P = 18$ and $A = 12$. In fact, all multiples of H lead to this solution or $\frac{b}{P}$ undefined.

4. Rank Calculations

There is, currently, no known guaranteed method to determine the rank r. We can estimate r very well, computationally, if we assume the Birch and Swinnerton-Dyer conjecture [6]. We performed the calculations using some home-grown software, with the Pari-gp package for the multiple-precision calculations. The results for $21 \leq n \leq 99$ are shown in Table 1.

TABLE 1
Ranks of $E_n, n = 21, \ldots, 99$

n	0	1	2	3	4	5	6	7	8	9
20+	–	1	0	0	1	0	1	0	1	0
30+	1	2	0	1	0	1	1	1	0	1
40+	0	0	2	2	0	1	0	2	0	0
50+	1	1	1	0	0	1	1	0	2	0
60+	1	1	1	2	1	0	1	1	0	0
70+	0	1	0	0	2	1	1	1	0	2
80+	0	1	0	1	0	1	1	0	2	0
90+	1	2	0	1	1	1	1	0	1	1

We can see that the curve from the $(3, 4, 5)$ triangle with $n = 24$ has rank 1, but the $(5, 5, 8)$ triangle has a curve with rank 0, showing that this is the only triangle giving $n = 27$.

For those curves with rank 1, a by-product of the Birch and Swinnerton-Dyer computations is an estimate for the height of the generator of the rational points of infinite order. The height essentially gives an idea of the sizes of the numerators and denominators of the u coordinates. The largest height encountered was 10.25 for $n = 83$ (the height normalisation used is the one used by Silverman [4]).

All the heights computed are small enough that we could compute the generators fairly easily, again using some simple software. From the generators, we derive the list of triangles in Table 2.

5. Geometry of $0 < \frac{b}{P} < \frac{1}{2}$

The sharp-eyed reader will have noticed that several values of n, which have positive rank in Table 1, do not give a triangle in Table 2, despite generators being found To explain this, we need to consider the geometric implications on (u, v) from the bounds $0 < \frac{b}{P} < \frac{1}{2}$, or

$$0 < \frac{n(u - 64) + v}{4nu} < \frac{1}{2} \tag{8}$$

Consider first $u > 0$. Then $\frac{b}{P} > 0$ when $v > 64n - nu$. The line $v = 64n - nu$, only meets E_n at $u = 0$, and the negative gradient shows that $\frac{b}{P} > 0$ when we take points on the upper part of the curve. To have $\frac{b}{P} < \frac{1}{2}$, we need $v < nu + 64n$. The line $v = nu + 64n$ has an intersection of multiplicity 3 at $u = 0$, so never meets E_n again. Thus $v < nu + 64n$ only on the lower part of E_n for $u > 0$. Thus, we cannot have $0 < \frac{b}{P} < \frac{1}{2}$ for any points with $u > 0$.

Now consider $u < 0$. Then, for $\frac{b}{P} > 0$ we need $v < 64n - nu$. The negative gradient and single intersection show that this holds for all points on E_n with $u < 0$. For $\frac{b}{P} < \frac{1}{2}$, we need $v > nu + 64n$. This line goes through $(0, 64n)$ on the curve and crosses the u-axis when $u = -64$, which we saw earlier lies strictly between the egg and the infinite component. Since $(0, 64n)$ is the only intersection we must have the line above the infinite component of E_n when $u < 0$ but below the egg.

TABLE 2
Triangles for $21 \leq n \leq 99$

n	a	b	c
21	15	14	13
28	35	34	15
31	85	62	39
35	97	78	35
39	37	26	15
43	56498	31695	29197
47	4747	3563	1560
51	149	85	72
55	157	143	30
58	85	60	29
62	598052	343383	275935
66	65	34	33
75	74	51	25
77	1435	2283	902
81	26	25	3
88	979	740	261
93	2325	2290	221
98	2307410	2444091	255319

n	a	b	c
24	5	4	3
30	13	12	5
33	30	25	11
36	17	10	9
42	20	15	7
45	41	40	9
50	1018	707	375
52	5790	4675	1547
56	41	28	15
60	29	25	6
63	371	250	135
74	740	723	91
76	47575	43074	7163
79	1027	1158	185
85	250	221	39
91	1625	909	742
95	24093	29582	6175
99	97	90	11

Thus, $0 < \frac{b}{P} < \frac{1}{2}$ only on the egg where $u < -64$. The results in Table 2 come from generators satisfying this condition.

It might be thought that forming integer multiples of generators and possibly adding the torsion points could resolve this. This is not the case, due to the closed nature of the egg. If a line meets the egg and is not a tangent to the egg, then it enters the egg and must exit the egg. Thus any line has a double intersection with the egg.

So, if we add a point on the infinite component to either torsion point, also on the infinite component, we must have the third intersection on the infinite component. Similarly doubling a point on the infinite component must lead to a point on the infinite component. So, if no generator lies on the egg, there will never be a point on the egg, and so no real-life triangle will exist.

We can generate other triangles for a value of n by taking multiples of the generator. Using the same arguments as before, it is clear that a generator G on the egg has $2G$ on the infinite component but $3G$ must lie on the egg. So, for $n = 24$, the curve $E_{24} : v^2 = u^3 + 576u^2 + 73728u + 2359296$ has $G = (-384, 1536)$, hence $2G = (768, -29184)$ and $3G = \left(-\frac{2240}{9}, \frac{55808}{27}\right)$, which leads to the triangle $(287, 468, 505)$ where $P = 1260$ and $A = 66150$.

References

[1] G. Baloglou and M. Helfgott, Angles, area and perimeter caught in a cubic, *Forum. Geom.*, 8 (2008) 13–25.

[2] L. Markov, Heronian triangles whose areas are integer multiples of their perimeters, *Forum Geom.*, 7 (2007) 129–135.

[3] L. J. Mordell, *Diophantine Equations*, Academic Press, New York, 1968.
[4] J. H. Silverman, Computing rational points on rank 1 elliptic curves via L-series and canonical heights, *Math. Comp.* 68 (1999) 835–858.
[5] J. H. Silverman and J. Tate, *Rational Points on Elliptic Curves*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1992.
[6] A. Wiles, *The Birch and Swinnerton-Dyer Conjecture*, available at the Clay Mathematics website `www.claymath.org`.

Allan J. MacLeod: Department of Mathematics and Statistics, University of the West of Scotland, High St., Paisley, Scotland. PA1 2BE.
E-mail address: `allan.macleod@uws.ac.uk`