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Characterizations of a Tangential Quadrilateral

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Abstract. In this paper we will present several relations about the tangential quadrilaterals; among these, we have that the quadrilateral $ABCD$ is tangential if and only if the following equality

$$\frac{1}{d(O,AB)}+\frac{1}{d(O,CD)}=\frac{1}{d(O,BC)}+\frac{1}{d(O,DA)}$$

holds, where O is the point where the diagonals of convex quadrilateral $ABCD$ meet. This is equivalent to Wu's Theorem.

A tangential quadrilateral is a convex quadrilateral whose sides all tangent to a circle inscribed in the quadrilateral. [1] In a tangential quadrilateral, the four angle bisectors meet at the center of the inscribed circle. Conversely, a convex quadrilateral in which the four angle bisectors meet at a point must be tangential. A necessary and sufficient condition for a convex quadrilateral to be tangential is that its two pairs of opposite sides have equal sums (see [1, 2, 4]). In [5], Marius Iosifescu proved that a convex quadrilateral $ABCD$ is tangential if and only if

$$\tan\frac{x}{2}\cdot\tan\frac{z}{2}=\tan\frac{y}{2}\cdot\tan\frac{w}{2},$$

where x, y, z, w are the measures of the angles ABD, ADB, BDC, and DBC respectively (see Figure 1). In [3], Wu Wei Chao gave another characterization of tangential quadrilaterals. The two diagonals of any convex quadrilateral divide the quadrilateral into four triangles. Let r_1, r_2, r_3, r_4, in cyclic order, denote the radii of the circles inscribed in each of these triangles (see Figure 2). Wu found that the quadrilateral is tangential if and only if

$$\frac{1}{r_1}+\frac{1}{r_3}=\frac{1}{r_2}+\frac{1}{r_4}.$$

In this paper we find another characterization (Theorem 1 below) of tangential quadrilaterals. This new characterization is shown to be equivalent to Wu's condition and others (Proposition 2).

Consider a convex quadrilateral $ABCD$ with diagonals AC and BD intersecting at O. Denote the lengths of the sides AB, BC, CD, DA by a, b, c, d respectively.

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[1]Tangential quadrilateral are also known as circumscriptible quadrilaterals, see [2, p.135].

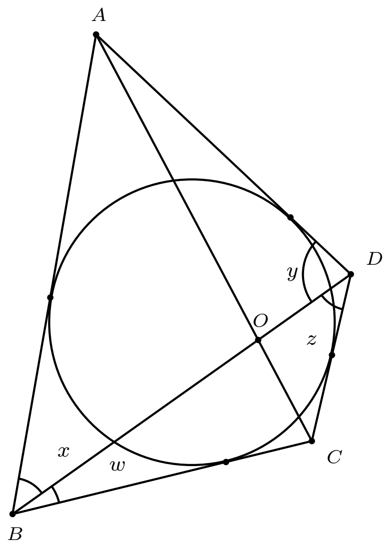

Figure 1

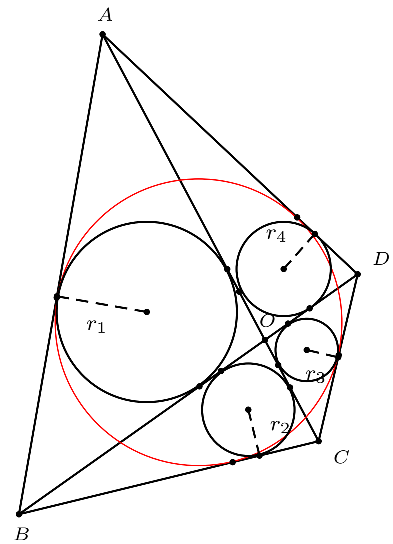

Figure 2

Theorem 1. *A convex quadrilateral $ABCD$ with diagonals intersecting at O is tangential if and only if*

$$\frac{1}{d(O,AB)}+\frac{1}{d(O,CD)}=\frac{1}{d(O,BC)}+\frac{1}{d(O,DA)}, \tag{1}$$

where $d(O,AB)$ is the distance from O to the line AB etc.

Proof. We first express (1) is an alternative form. Consider the projections M, N, P and Q of O on the sides AB, BC, CD, DA respectively.

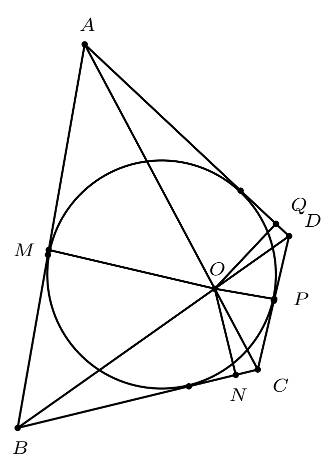

Figure 3

Let $AB = a$, $BC = b$, $CD = c$, $DA = d$. It is easy to see

$$\frac{OM}{d(C,AB)} = \frac{AO}{AC} = \frac{OQ}{d(C,AD)},$$
$$\frac{OM}{d(D,AB)} = \frac{BO}{BD} = \frac{ON}{d(D,BC)},$$
$$\frac{ON}{d(A,BC)} = \frac{OC}{AC} = \frac{OP}{d(A,DC)}.$$

This means

$$\frac{OM}{b\sin B} = \frac{OQ}{c\sin D}, \quad \frac{OM}{d\sin A} = \frac{ON}{c\sin C}, \quad \frac{ON}{a\sin B} = \frac{OP}{d\sin D}.$$

The relation (1) becomes

$$\frac{1}{OM} + \frac{1}{OP} = \frac{1}{ON} + \frac{1}{OQ},$$

which is equivalent to

$$1 + \frac{OM}{OP} = \frac{OM}{ON} + \frac{OM}{OQ},$$

or

$$1 + \frac{a\sin A\sin B}{c\sin C\sin D} = \frac{d\sin A}{c\sin C} + \frac{b\sin B}{c\sin D}.$$

Therefore (1) is equivalent to

$$a\sin A\sin B + c\sin C\sin D = b\sin B\sin C + d\sin D\sin A. \qquad (2)$$

Now we show that $ABCD$ is tangential if and only if (2) holds.

(⇒) If the quadrilateral $ABCD$ is tangential, then there is a circle inscribed in the quadrilateral. Let r be the radius of this circle. Then

$$a = r\left(\cot\frac{A}{2} + \cot\frac{B}{2}\right), \qquad b = r\left(\cot\frac{B}{2} + \cot\frac{C}{2}\right),$$
$$c = r\left(\cot\frac{C}{2} + \cot\frac{D}{2}\right), \qquad d = r\left(\cot\frac{D}{2} + \cot\frac{A}{2}\right).$$

Hence,

$$\begin{aligned}
a\sin A\sin B &= r\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\cdot 4\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2}\\
&= 4r\left(\cos\frac{A}{2}\sin\frac{B}{2}+\cos\frac{B}{2}\sin\frac{A}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2}\\
&= 4r\sin\frac{A+B}{2}\cos\frac{A}{2}\cos\frac{B}{2}\\
&= 4r\sin\frac{C+D}{2}\cos\frac{A}{2}\cos\frac{B}{2}\\
&= 4r\left(\cos\frac{D}{2}\sin\frac{C}{2}+\cos\frac{C}{2}\sin\frac{D}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2}\\
&= 4r\left(\tan\frac{C}{2}+\tan\frac{D}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{D}{2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
b\sin B\sin C &= 4r\left(\tan\frac{D}{2}+\tan\frac{A}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{D}{2},\\
c\sin C\sin D &= 4r\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{D}{2},\\
d\sin D\sin A &= 4r\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{D}{2}.
\end{aligned}$$

From these relations it is clear that (2) holds.

($\Leftarrow$) We assume (2) and $ABCD$ not tangential. From these we shall deduce a contradiction.

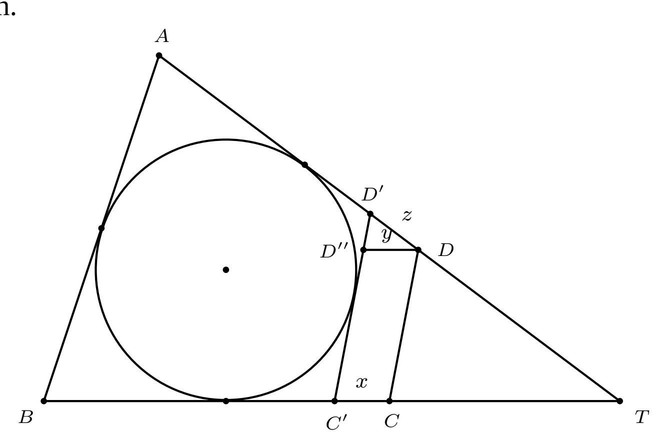

Figure 4.

Case 1. Suppose the opposite sides of $ABCD$ are not parallel.

Let T be the intersection of the lines AD and BC. Consider the incircle of triangle ABT (see Figure 4). Construct a parallel to the side DC which is tangent to the circle, meeting the sides BC and DA at C' and D' respectively. Let $BC' =$

b', $C'D' = c'$, $D'A = d'$, $C'C = x$, $D''D' = y$, and $D'D = z$, and where D'' is the point on $C'D'$ such that $C'CDD''$ is a parallelogram. Note that

$$b = b' + x, \quad c = c' - y, \quad d = d' + z.$$

Since the quadrilateral $ABC'D'$ is tangential, we have

$$a \sin A \sin B + c' \sin C \sin D = b' \sin B \sin C + d' \sin D \sin A. \qquad (3)$$

Comparison of (2) and (3) gives

$$a \sin A \sin B + c \sin C \sin D = b \sin B \sin C + d \sin D \sin A,$$

we have

$$-y \sin C \sin D = x \sin B \sin C + z \sin D \sin A.$$

This is a contradiction since x, y, z all have the same sign, [2] and the trigonometric ratios are all positive.

Case 2. Now suppose $ABCD$ has a pair of parallel sides, say AD and BC. Consider the circle tangent to the sides AB, BC and DA (see Figure 5).

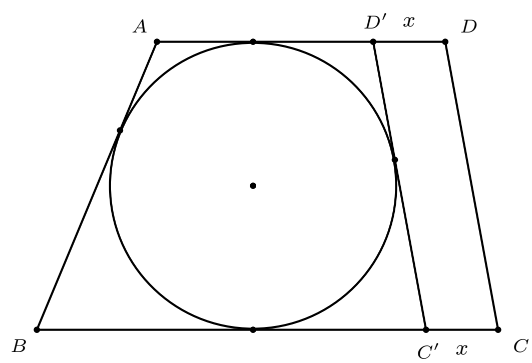

Figure 5.

Construct a parallel to DC, tangent to the circle, and intersecting BC, DA at C' and D' respectively. Let $C'C = D'D = x$, $BC' = b'$, and $D'A = d'$. [3] Clearly, $b' = b - x$, $d = d' + x$, and $C'D' = CD = c$. Since the quadrilateral $ABC'D'$ is tangential, we have

$$a \sin A \sin B + c \sin C \sin D = b' \sin B \sin C + d' \sin D \sin A. \qquad (4)$$

Comparing this with (2), we have $x(\sin B \sin C + \sin D \sin A) = 0$. Since $x \neq 0$, $\sin A = \sin B$ and $\sin C = \sin D$, this reduces to $2 \sin A \sin C = 0$, a contradiction. □

Proposition 2. *Let O be the point where the diagonals of the convex quadrilateral $ABCD$ meet and r_1, r_2, r_3, and r_4 respectively the radii of the circles inscribed in the triangles AOB, BOC, COD and DOA respectively. The following statements are equivalent:*

[2]In Figure 4, the circle does not intersect the side CD. In case it does, we treat x, y, z as negative.

[3]Again, if the circle intersects CD, then x is regarded as negative.

(a) $\frac{1}{r_1} + \frac{1}{r_3} = \frac{1}{r_2} + \frac{1}{r_4}$.
(b) $\frac{1}{d(O,AB)} + \frac{1}{d(O,CD)} = \frac{1}{d(O,BC)} + \frac{1}{d(O,DA)}$.
(c) $\frac{a}{\Delta AOB} + \frac{c}{\Delta COD} = \frac{b}{\Delta BOC} + \frac{d}{\Delta DOA}$.
(d) $a \cdot \Delta COD + c \cdot \Delta AOB = b \cdot \Delta DOA + d \cdot \Delta BOC$.
(e) $a \cdot OC \cdot OD + c \cdot OA \cdot OB = b \cdot OA \cdot OD + d \cdot OB \cdot OC$.

Proof. (a) ⇔ (b). The inradius of a triangle is related to the altitudes by the simple relation

$$\frac{1}{r} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}.$$

Applying this to the four triangles AOB, BOC, COD, and DOA, we have

$$\frac{1}{r_1} = \frac{1}{d(O,AB)} + \frac{1}{d(A,BD)} + \frac{1}{d(B,AC)},$$
$$\frac{1}{r_2} = \frac{1}{d(O,BC)} + \frac{1}{d(C,BD)} + \frac{1}{d(B,AC)},$$
$$\frac{1}{r_3} = \frac{1}{d(O,CD)} + \frac{1}{d(C,BD)} + \frac{1}{d(D,AC)},$$
$$\frac{1}{r_4} = \frac{1}{d(O,DA)} + \frac{1}{d(A,BD)} + \frac{1}{d(D,AC)}.$$

From these the equivalence of (a) and (b) is clear.

(b) ⇔ (c) is clear from the fact that $\frac{1}{d(O,AB)} = \frac{a}{a \cdot d(O,AB)} = \frac{a}{2\Delta AOB}$ etc.

The equivalence of (c), (d) and (e) follows from follows from

$$\Delta AOB = \frac{1}{2} \cdot OA \cdot OB \cdot \sin\varphi$$

etc., where φ is the angle between the diagonals. Note that

$$\Delta AOB \cdot \Delta COD = \Delta BOC \cdot \Delta DOA = \frac{1}{4} \cdot OA \cdot OB \cdot OC \cdot OD \cdot \sin^2\varphi.$$

□

References

[1] T. Andreescu and B. Enescu, *Mathematical Olympiad Treasures*, Birkhäuser, Boston, 2004.

[2] N. Altshiller-Court, *College Geometry*, second edition, Barnes and Noble, 1952; Dover reprint, 2007.

[3] W. C. Wu and P. Simeonov, Problem 10698, *Amer. Math. Monthly*, 105 (1998) 995; solution, 107 (2000) 657–658.

[4] P. Yiu, Euclidean Geometry, Florida Atlantic University Lecture Notes, 1998, available at `http://math.fau.edu/Yiu/Geometry.html`.

[5] M. Iosifescu, Problem 1421, *The Mathematical Gazette*, (in Romanian) no.11, 1954.

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