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Class Preserving Dissections of Convex Quadrilaterals

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Abstract. Given a convex quadrilateral Q having a certain property $\mathcal{P}$, we are interested in finding a dissection of Q into a finite number of smaller convex quadrilaterals, each of which has property $\mathcal{P}$ as well. In particular, we prove that every cyclic, orthodiagonal, or circumscribed quadrilateral can be dissected into cyclic, orthodiagonal, or circumscribed quadrilaterals, respectively. The problem becomes much more interesting if we restrict our study to a particular type of partition we call *grid dissection*.

1. Introduction

The following problem represents the starting point and the motivation of this paper.

Problem. Find all convex polygons which can be dissected into a finite number of pieces, each similar to the original one, but not necessarily congruent.

It is easy to see that all triangles and parallelograms have this property (see e.g. [1, 7]).

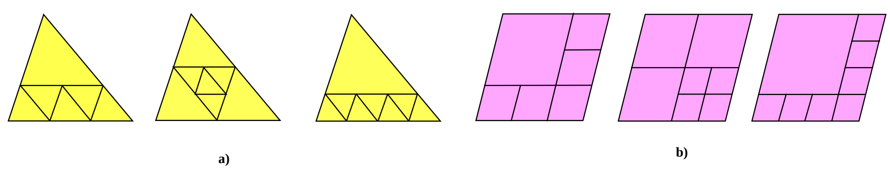

Figure 1. (a) Triangle dissection into similar triangles.
(b) Parallelogram dissection into similar parallelograms.

Indeed, every triangle can be partitioned into 6, 7 or 8 triangles, each similar to the initial one (see Figure 1 a). Simple inductive reasoning shows that for every $k \geq 6$, any triangle T can be dissected into k triangles similar to T. An analogous statement is true for parallelograms (see Figure 1 b). Are there any other polygons besides these two which have this property?

The origins of Problem 1 can be traced back to an early paper of Langford [10]. More then twenty years later, Golomb [8] studied the same problem without notable success. It was not until 1974 when the first significant results were published by Valette and Zamfirescu.

Theorem 1 (Valette and Zamfirescu, [13]). *Suppose a given convex polygon P can be dissected into four congruent tiles, each of which similar to P. Then P is either a triangle, a parallelogram or one of the three special trapezoids shown in Figure 2 below.*

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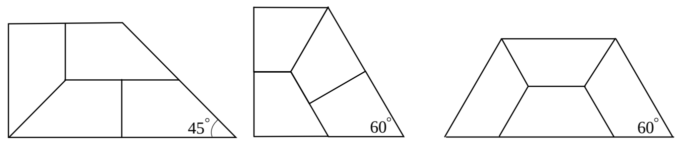

Figure 2. Trapezoids which can be partitioned into four congruent pieces.

Notice that the hypothesis of the above theorem is much more restrictive: the number of pieces must be exactly four and the small polygons must all be congruent to each other, not only similar. However, as of today, the convex polygons presented in figures 1 and 2 are the only known solutions to the more general problem 1.

From a result of Bleicher [2], it is impossible to dissect a convex n-gon (a convex polygon with n vertices) into a finite number of convex n-gons if $n \geq 6$. The same result was proved by Bernheim and Motzkin [3] using slightly different techniques.

Although any convex pentagon can be partitioned into any number $k \geq 6$ of convex pentagons, a recent paper by Ding, Schattschneider and Zamfirescu [4] shows that it is impossible to dissect a convex pentagon into similar replicas of itself.

Given the above observations, it follows that for solving problem 1 we can restrict ourselves to convex quadrilaterals. It is easy to prove that a necessary condition for a quadrilateral to admit a dissection into similar copies of itself is that the measures of its angles are linearly dependent over the integers. Actually, a stronger statement holds true: if the angles of a convex quadrilateral Q do not satisfy this dependence condition, then Q cannot be dissected into a finite number of smaller similar convex polygons which are not necessarily similar to Q (for a proof one may consult [9]). Nevertheless, in spite of all the above simplifications and renewed interest in the geometric dissection topic (see e. g. [6, 12, 15]), problem 1 remains open.

2. A Related Dissection Problem

Preserving similarity under dissection is difficult: although all triangles have this property, there are only a handful of known quadrilaterals satisfying this condition (parallelograms and some special trapezoids), while no n-gon can have this property if $n \geq 5$. In the sequel, we will try to examine what happens if we weaken the similarity requirement.

Problem. Suppose that a given polygon P has a certain property $\mathcal{C}$. Is it possible to dissect P into smaller polygons, each having property $\mathcal{C}$ as well?

For instance, suppose $\mathcal{C}$ means "convex polygon with n sides". As we have mentioned in the previous section, in this particular setting Problem 2 has a positive answer if $3 \leq n \leq 5$ and a negative answer for all $n \geq 6$. Before we proceed we need the following:

Definition. a) A convex quadrilateral is said to be *cyclic* if there exists a circle passing through all of its vertices.
b) A convex quadrilateral is said to be *orthodiagonal* if its diagonals are perpendicular.
c) A convex quadrilateral is said to be *circumscribed* if there exists a circle tangent to all of its sides.
d) A convex quadrilateral is said to be a *kite* if it is both orthodiagonal and circumscribed.

The following theorem provides characterizations for all of the quadrilaterals defined above and will be used several times throughout the remainder of the paper.

Theorem 2. *Let $ABCD$ be a convex quadrilateral.*
(a) *$ABCD$ is cyclic if and only if opposite angles are supplementary – say, $\angle A + \angle C = 180°$.*
(b) *$ABCD$ is orthodiagonal if and only if the sum of squares of two opposite sides is equal to the sum of the squares of the remaining opposite sides – that is, $AB^2 + CD^2 = AD^2 + BC^2$.*
(c) *$ABCD$ is circumscribed if and only if the two pairs of opposite sides have equal total lengths – that is, $AB + CD = AD + BC$.*
(d) *$ABCD$ is a kite if and only if (after an eventual relabeling) $AB = BC$ and $CD = DA$.*

A comprehensive account regarding cyclic, orthodiagonal and circumscribed quadrilaterals and their properties, including proofs of the above theorem, can be found in the excellent collection of geometry notes [14]. An instance of Problem 2 we will investigate is the following:

Problem. Is it true that every cyclic, orthodiagonal or circumscribed quadrilateral can be dissected into cyclic, orthodiagonal or circumscribed quadrilaterals, respectively?

It has been shown in [1] and [11] that every cyclic quadrilateral can be dissected into **four** cyclic quadrilaterals two of which are isosceles trapezoids (see Figure 3 a).

Another result is that every cyclic quadrilateral can be dissected into **five** cyclic quadrilaterals one of which is a rectangle (see Figure 3 b). This dissection is based on the following property known as *The Japanese Theorem* (see [5]).

Theorem 3. *Let $ABCD$ be a cyclic quadrilateral and let M, N, P and Q be the incenters of triangles ABD, ABC, BCD and ACD, respectively. Then $MNPQ$ is a rectangle and quadrilaterals $AMNB$, $BNPC$, $CPQD$ and $DQMA$ are cyclic* (see Figure 3).

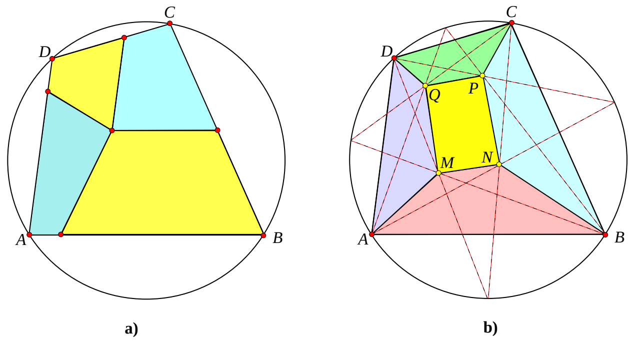

Figure 3. (a) Cyclic quadrilateral = 2 isosceles trapezoids + 2 cyclic quadrilaterals.
(b) Cyclic quadrilateral = one rectangle + four cyclic quadrilaterals.

Since every isosceles trapezoid can be dissected into an arbitrary number of isosceles trapezoids, it follows that every cyclic quadrilateral can be dissected into k cyclic quadrilaterals, for every $k \geq 4$.

It is easy to dissect an orthodiagonal quadrilateral into four smaller orthodiagonal ones.

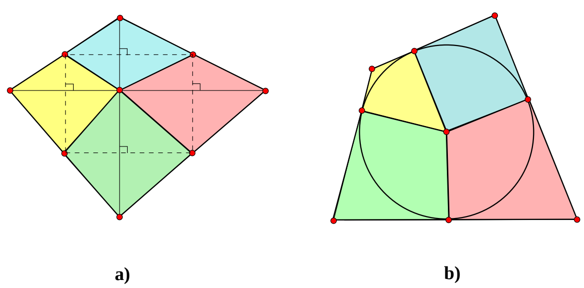

Figure 4. (a) Orthodiagonal quadrilateral = four orthodiagonal quadrilaterals.
(b) Circumscribed quadrilateral = four circumscribed quadrilaterals.

Consider for instance the quadrilaterals whose vertex set consists of one vertex of the initial quadrilateral, the midpoints of the sides from that vertex and the intersection point of the diagonals (see Figure 4 a). It is easy to prove that each of these quadrilaterals is orthodiagonal.

A circumscribed quadrilateral can be dissected into four quadrilaterals with the same property by simply taking the radii from the incenter to the tangency points (see Figure 4 b).

Actually, it is easy to show that each of these smaller quadrilaterals is not only circumscribed but cyclic and orthodiagonal as well.

The above discussion provides a positive answer to problem 2. In fact, much more is true.

Theorem 4 (Dissecting arbitrary polygons)**.** *Every convex n-gon can be partitioned into $3\,(n-2)$ cyclic kites* (see Figure 5).

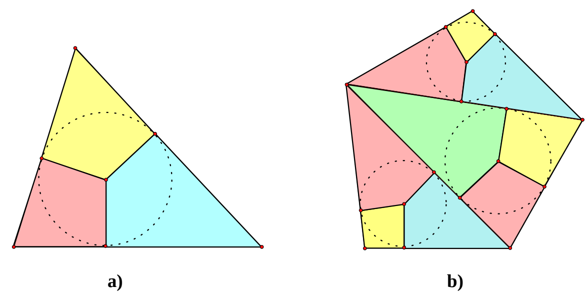

Figure 5. Triangle = three cyclic kites; Pentagon = nine cyclic kites.

Proof. Notice first that every triangle can be dissected into three cyclic kites by dropping the radii from the incenter to the tangency points (see Figure 5 a). Partition the given n-gon into triangles. For instance, one can do this by drawing all the diagonals from a certain vertex. We obtain a triangulation consisting of $n-2$ triangles. Dissect then each triangle into cyclic kites as indicated in Figure 5 b). $\square$

3. Grid Dissections of Convex Quadrilaterals

We have seen that the construction used in theorem 4 renders problem 2 almost trivial. The problem becomes much more challenging if we do restrict the type of dissection we are allowed to use. We need the following

Definition. Let $ABCD$ be a convex quadrilateral and let m and n be two positive integers. Consider two sets of segments $\mathcal{S} = \{s_1, s_2, \ldots, s_{m-1}\}$ and $\mathcal{T} = \{t_1, t_2, \ldots, t_{n-1}\}$ with the following properties:
a) If $s \in \mathcal{S}$ then the endpoints of s belong to the sides AB and CD. Similarly, if $t \in \mathcal{T}$ then the endpoints of t belong to the sides AD and BC.
b) Every two segments in $\mathcal{S}$ are pairwise disjoint and the same is true for the segments in $\mathcal{T}$.
We then say that segments $s_1, s_2, \ldots, s_{m-1}, t_1, t_2, \ldots, t_{n-1}$ define an m-by-n *grid dissection* of $ABCD$ (see Figure 6).

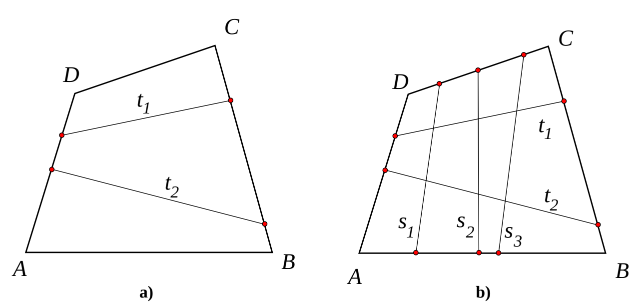

Figure 6. A 3-by-1 and a 3-by-4 grid dissection of a convex quadrilateral

The really interesting problem is the following:

Problem. Is it true that every cyclic, orthodiagonal or circumscribed quadrilateral can be partitioned into cyclic, orthodiagonal, or circumscribed quadrilaterals,respectively, **via a grid dissection?** Such dissections shall be referred to as *class preserving grid dissections*, or for short *CPG* dissections (or CPG partitions).

3.1. *Class Preserving Grid Dissections of Cyclic Quadrilaterals.* In this section we study whether cyclic quadrilaterals have class preserving grid dissections. We start with the following
Question. Under what circumstances does a cyclic quadrilateral admit a 2-by-1 grid dissection into cyclic quadrilaterals? What about a 2-by-2 grid dissection with the same property?

The answer can be readily obtained after a straightforward investigation of the sketches presented in Figure 7.

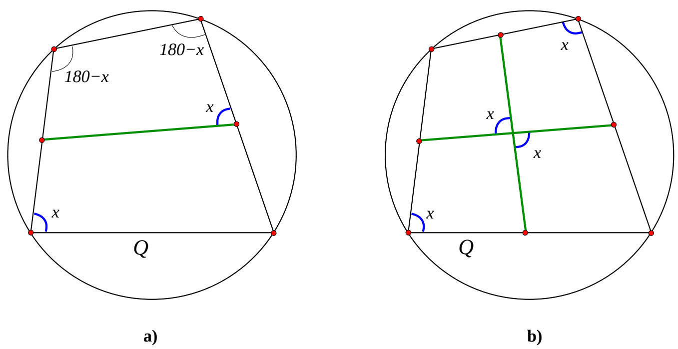

Figure 7. (a) 2-by-1 CPG dissection of cyclic Q exists iff $Q =$ trapezoid.
(b) 2-by-2 CPG dissection of cyclic Q exists iff $Q =$ rectangle.

A quick analysis of the angles reveals that a 2-by-1 CPG partition is possible if only if the initial cyclic quadrilateral is an isosceles trapezoid - see Figure 7 a). A similar reasoning leads to the conclusion that a 2-by-2 CPG partition exists if and only if the original quadrilateral is a rectangle – Figure 7 b). These observations can be easily extended to the following:

Theorem 5. *Suppose a cyclic quadrilateral Q has an m-by-n grid partition into mn cyclic quadrilaterals. Then:*
a) If m and n are both even, Q is necessarily a rectangle.
b) If m is odd and n is even, Q is necessarily an isosceles trapezoid.

We leave the easy proof for the reader. It remains to see what happens if both m and n are odd. The next two results show that in this case the situation is more complex.

Theorem 6 (A class of cyclic quadrilaterals which have 3-by-1 CPG dissections)**.** *Every cyclic quadrilateral all of whose angles are greater than* $\arccos \frac{\sqrt{5}-1}{2} \approx 51.83°$ *admits a* 3*-by-*1 *grid dissection into three cyclic quadrilaterals.*

Proof. If $ABCD$ is an isosceles trapezoid, then any two segments parallel to the bases will give the desired dissection. Otherwise, assume that $\angle D$ is the largest angle (a relabeling of the vertices may be needed). Since $\angle B+\angle D = \angle A+\angle C = 180^\circ$ it follows that $\angle B$ is the smallest angle of $ABCD$. We therefore have:

$$\angle B < \min\{\angle A,\, \angle C\} \le \max\{\angle A,\, \angle C\} < \angle D. \tag{1}$$

Denote the measures of the arcs $\widehat{AB}$, $\widehat{BC}$, $\widehat{CD}$ and $\widehat{DA}$ on the circumcircle of $ABCD$ by $2a$, $2b$, $2c$ and $2d$ respectively (see Figure 8 a). Inequalities (1) imply that $c+d < \min\{b+c,\, a+d\} \le \max\{b+c,\, a+d\} < a+b$, that is, $c < a$ and $d < b$.

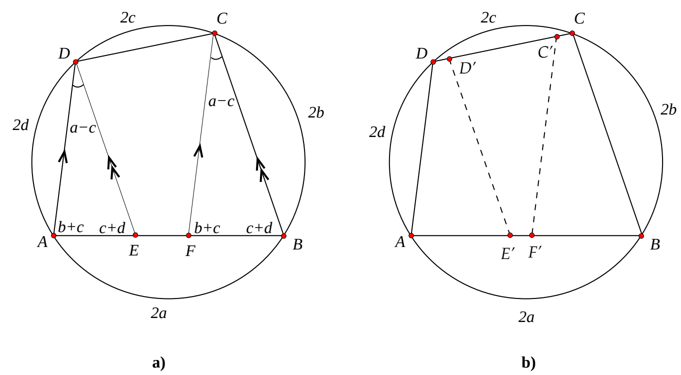

Figure 8. (a) $DE \parallel BC$, $CF \parallel AD$, E and F between A and B due to $c < a$.
(b) 3-by-1 grid dissection into cyclic quads if DE and CF do not intersect.

Through vertex D construct a segment $DE \parallel BC$ with E on line AB. Since $c < a$, point E is going to be between A and B. Similarly, through vertex C construct a segment $CF \parallel AD$ with F on AB. As above, since $c < a$, point F will lie between A and B.

If segments DE and CF do not intersect then a 3-by-1 grid dissection of $ABCD$ into cyclic quadrilaterals can be obtained in the following way:
Choose two points C' and D' on side CD, such that C' is close to C and D' is close to D. Construct $D'E' \parallel DE$ and $C'F' \parallel CF$ as shown in figure 8 b). Since segments DE and CF do not intersect it follows that for choices of C' and D' sufficiently close to C and D respectively, the segments $D'E'$ and $C'F'$ will not intersect. A quick verification shows that each of the three quadrilaterals into which $ABCD$ is dissected ($AE'D'D$, $D'E'F'C'$ and $C'F'BC$) is cyclic.

It follows that a sufficient condition for this grid dissection to exist is that points $A - E - F - B$ appear exactly in this order along side AB, or equivalently, $AE + BF < AB$.

The law of sines in triangle ADE gives that $AE\sin(c+d) = AD\sin(a-c)$ and since $AD = 2R\sin d$ we obtain

$$AE = \frac{2R\sin d\,\sin(a-c)}{\sin(c+d)}, \tag{2}$$

where R is the radius of the circumcircle of $ABCD$.

Similarly, using the law of sines in triangle BCF we have $BF\,\sin(b+c) = BC\,\sin(a-c)$ and since $BC = 2R\,\sin b$ it follows that

$$BF = \frac{2R\sin b\,\sin(a-c)}{\sin(b+c)}. \tag{3}$$

Using equations (2), (3) and the fact that $AB = 2R\,\sin a$, the desired inequality $AE + BF < AB$ becomes equivalent to

$$\begin{aligned} &\frac{\sin d\,\sin(a-c)}{\sin(c+d)} + \frac{\sin b\,\sin(a-c)}{\sin(b+c)} < \sin a \\ \Leftrightarrow\ &\frac{\sin d}{\sin(c+d)} + \frac{\sin(b+c-c)}{\sin(b+c)} < \frac{\sin(a-c+c)}{\sin(a-c)} \\ \Leftrightarrow\ &\frac{\sin d}{\sin(c+d)} + \cos c - \sin c\,\cot(b+c) < \cos c + \sin c\,\cot(a-c), \end{aligned}$$

and after using $a+b+c+d = 180^\circ$ and simplifying further,

$$AE + BF < AB \Leftrightarrow \sin(a-c)\,\sin(b+c)\,\sin d < \sin^2(c+d)\,\sin(c). \tag{4}$$

Recall that points E and F belong to AB as a result of the fact that $c < a$. A similar construction can be achieved using the fact that $d < b$.

Let $AG \parallel CD$ and $DH \parallel AB$ as shown in Figure 9 a). Since $d < b$, points G and H will necessarily belong to side BC. As in the earlier analysis, if segments AG and DH do not intersect, small parallel displacements of these segments will produce a 3-by-1 grid partition of $ABCD$ into 3 cyclic quadrilaterals: $ABG'A'$, $A'G'H'D''$ and $H'D''DC$ (see Figure 9 b).

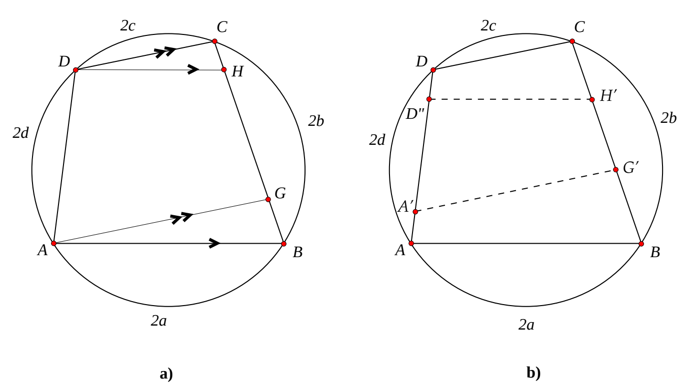

Figure 9. a) $AG \parallel CD$, $DH \parallel AB$, G and H between B and C since $d < b$
b) 3-by-1 CPG grid dissection if AG and DH do not intersect.

The sufficient condition for this construction to work is that points $B{-}G{-}H{-}C$ appear in this exact order along side BC, or equivalently, $BG + CH < BC$.

Using similar reasoning which led to relation (4) we obtain that

$$BG + CH < BC \Leftrightarrow \sin(b-d)\,\sin(a+d)\,\sin c < \sin^2(a+b)\,\sin d. \tag{5}$$

The problem thus reduces to proving that if $\min\{\angle A, \angle B, \angle C, \angle D\} > \arccos \frac{\sqrt{5}-1}{2}$ then at least one of the inequalities that appear in (4) and (5) will hold.
To this end, suppose none of these inequalities is true. We thus have:

$$\begin{aligned} \sin(a-c)\,\sin(b+c) &\geq \sin^2(c+d)\cdot\frac{\sin c}{\sin d} \qquad \text{and,} \\ \sin(b-d)\,\sin(a+d) &\geq \sin^2(a+b)\cdot\frac{\sin d}{\sin c}. \end{aligned}$$

Recall that $a+b+c+d = 180^\circ$ and therefore $\sin(a+d) = \sin(b+c)$ and $\sin(a+b) = \sin(c+d)$. Adding the above inequalities term by term we obtain

$$\begin{aligned} &\sin(b+c)\cdot(\sin(a-c)+\sin(b-d)) \geq \sin^2(c+d)\cdot\left(\frac{\sin c}{\sin d}+\frac{\sin c}{\sin d}\right) \\ \Rightarrow\ &\sin(b+c)\cdot 2\cdot\sin(90^\circ-c-d)\cdot\cos(90^\circ-b-d) \geq \sin^2(c+d)\cdot 2 \\ \Leftrightarrow\ &\sin(b+c)\cdot\sin(c+d)\cdot\cos(c+d) \geq \sin^2(c+d) \\ \Rightarrow\ &\cos(c+d) \geq 1-\cos^2(c+d) \\ \Rightarrow\ &\cos(c+d) = \cos(\angle B) \geq \frac{\sqrt{5}-1}{2}, \text{contradiction.} \end{aligned}$$

This completes the proof. Notice that the result is the best possible in the sense that $\arccos\frac{\sqrt{5}-1}{2} \approx 51.83^\circ$ cannot be replaced by a smaller value. Indeed, it is easy to check that a cyclic quad whose angles are $\arccos\frac{\sqrt{5}-1}{2}$, 90°, 90° and $180^\circ - \arccos\frac{\sqrt{5}-1}{2}$ does not have a 3-by-1 grid partition into cyclic quadrilaterals. □

The following result can be obtained as a corollary of Theorem 6.

Theorem 7. (A class of cyclic quadrilaterals which have 3-by-3 grid dissections) *Let $ABCD$ be a cyclic quadrilateral such that the measure of each of the arcs $\widehat{AB}$, $\widehat{BC}$, $\widehat{CD}$ and $\widehat{DA}$ determined by the vertices on the circumcircle is greater than 60°. Then $ABCD$ admits a 3-by-3 grid dissection into nine cyclic quadrilaterals.*

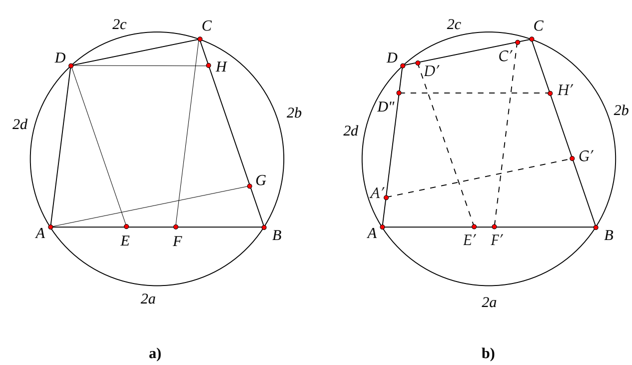

Figure 10. a) $DE \parallel BC$, $CF \parallel AD$, $AG \parallel CD$, $DH \parallel AB$
b) 3-by-3 grid dissection into nine cyclic quadrilaterals.

Proof. Notice that the condition regarding the arc measures is stronger than the requirement that all angles of $ABCD$ exceed 60°. We will use the same assumptions and notations as in Theorem 6. The idea is to overlay the two constructions in Theorem 6 (see Figure 10).

It is straightforward to check that each of the nine quadrilaterals shown in figure 10 b) is cyclic. The problem reduces to proving that $\min\{a,\, b,\, c,\, d\} > 30^\circ$ implies that both inequalities in (4) and (5) hold simultaneously. Due to symmetry it is sufficient to prove that (5) holds. Indeed,

$$\begin{aligned}
BG + CH < BC &\Leftrightarrow\ \sin(b-d)\,\sin(b+c)\,\sin c < \sin^2(c+d)\,\sin d \\
&\Leftrightarrow\ \cos(c+d) - \cos(2b+c-d) < \frac{2\,\sin^2(c+d)\,\sin d}{\sin c} \\
&\Leftarrow\ \cos(c+d) + 1 < \frac{2\sin^2(c+d)\sin d}{\sin c} \\
&\Leftrightarrow\ 2\cos^2\frac{c+d}{2}\sin c < 8\sin^2\frac{c+d}{2}\cos^2\frac{c+d}{2}\sin d \\
&\Leftrightarrow\ \sin c < 2\,\sin d\cdot(1-\cos(c+d)) \\
&\Leftrightarrow\ \sin c < 2\,\sin d - 2\,\sin d\,\cos(c+d) \\
&\Leftrightarrow\ \sin c < 2\,\sin d + \sin c - \sin(c+2d)) \\
&\Leftrightarrow\ \sin(c+2d) < 2\sin d \\
&\Leftarrow\ 1 < 2\sin d.
\end{aligned}$$

The last inequality holds true since we assumed $d > 30^\circ$. This completes the proof.

□

3.2. *Class Preserving Grid Dissections of Orthodiagonal Quadrilaterals.* It is easy to see that an orthodiagonal quadrilateral cannot have a 2-by-1 grid dissection into orthodiagonal quadrilaterals. Indeed, if say we attempt to dissect the quadrilateral $ABCD$ with a segment MP, where M is on AB and P is on CD, then the diagonals of $ADPM$ are forced to intersect in the interior of the right triangle AOD, preventing them from being perpendicular to each other (see Figure 11 a).

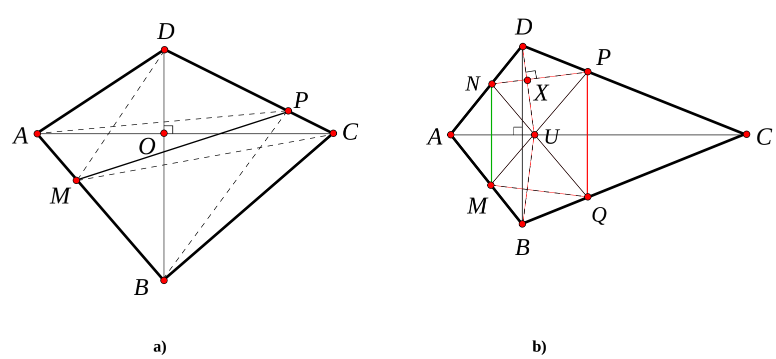

Figure 11. a) Orthodiagonal quadrilaterals have no 2-by-1 CPG dissections
b) A kite admits infinitely many 2-by-2 CPG dissections

The similar question concerning the existence of 2-by-2 CPG dissections turns out to be more difficult. We propose the following:

Conjecture 1. *An orthodiagonal quadrilateral has a* 2*-by-*2 *grid dissection into four orthodiagonal quadrilaterals if and only if it is a kite.*

The "only if" implication is easy to prove. We can show that every kite has infinitely many 2-by-2 CPG grid dissections. Indeed, let $ABCD$ be a kite ($AB = AD$ and $BC = CD$) and let $MN \parallel BD$ with M and N fixed points on sides AB and respectively AD. Consider then a variable segment $PQ \parallel BD$ as shown in figure 11 b). Denote $U = NQ \cap MP$; due to symmetry $U \in AC$. Consider the grid dissection generated by segments MP and NQ. Notice that quadrilaterals $ANUM$ and $CPUQ$ are orthodiagonal independent of the position of PQ. Also, quadrilaterals $DNUP$ and $BMUQ$ are congruent and therefore it is sufficient to have one of them be orthodiagonal.

Let point P slide along CD. If P is close to vertex C, it follows that Q and U are also close to C and therefore the measure of angle $\angle DXN$ is arbitrarily close to the measure of $\angle DCN$, which is acute. On the other hand, when P is close to vertex D, Q is close to B and the angle $\angle DXN$ becomes obtuse.

Since the measure of $\angle DXN$ depends continuously on the position of point P it follows that for some intermediate position of P on CD we will have $\angle DXN = 90°$. For this particular choice of P both $DNUP$ and $BMUQ$ are orthodiagonal. This proves the "only if" part of the conjecture.

Extensive experimentation with Geometer's Sketchpad strongly suggests the direct statement also holds true. We used MAPLE to verify the conjecture in several particular cases - for instance, the isosceles orthodiagonal trapezoid with base lengths of 1 and $\sqrt{7}$ and side lengths 2 does not admit a 2-by-2 dissection into orthodiagonal quadrilaterals.

3.3. *Class Preserving Grid Dissections of Circumscribed Quadrilaterals.* After the mostly negative results from the previous sections, we discovered the following surprising result.

Theorem 8. *Every circumscribed quadrilateral has a* 2*-by-*2 *grid dissection into four circumscribed quadrilaterals.*

Proof. (Sketch) This is in our opinion a really unexpected result. It appears to be new and the proof required significant amounts of inspiration and persistence. We approached the problem analytically and used MAPLE extensively to perform the symbolic computations. Still, the problem presented great challenges, as we will describe below.

Let $MNPQ$ be a circumscribed quadrilateral with incenter O. With no loss of generality suppose the incircle has unit radius. Let O_i, $1 \leq i \leq 4$ denote projections of O onto the sides as shown in Figure 12 a). Denote the angles $\angle O_4OO_1 = 2a$, $\angle O_1OO_2 = 2b$, $\angle O_2OO_3 = 2c$ and $\angle O_3OO_4 = 2d$. Clearly, $a + b + c + d = 180°$ and $\max\{a, b, c, d\} < 90°$. Consider a coordinate system centered at O such that the coordinates of O_4 are $(1, 0)$.

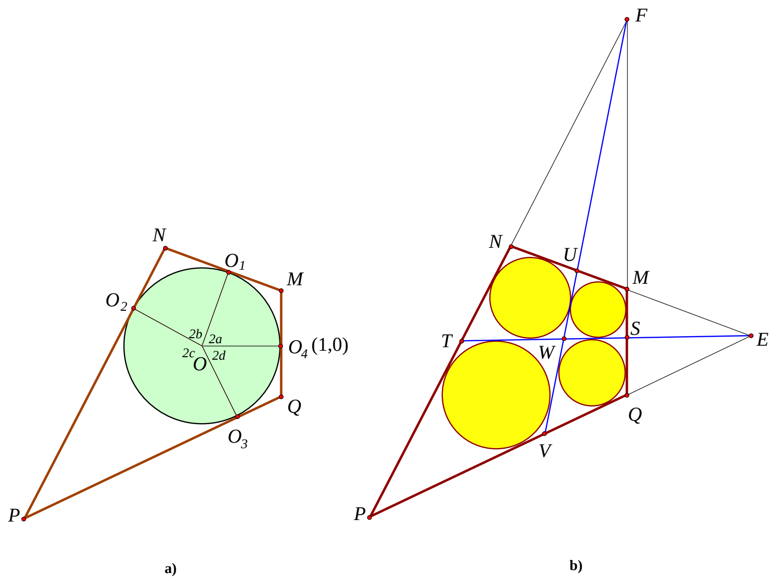

Figure 12. a) A circumscribed quadrilateral
b) Attempting a 2-by-2 CPG dissection with lines through E and F

We introduce some more notation: $\tan a = A$, $\tan b = B$, $\tan c = C$, $\tan d = D$. Notice that quantities A, B, C and D are not independent. Since $a + b + c + d = 180°$ it follows that $A+B+C+D = ABC+ABD+ACD+BCD$. Moreover, since $\max\{a,\ b,\ c,\ d\} < 90°$ we have that A, B, C and D are all positive.

It is now straightforward to express the coordinates of the vertices M, N, P and Q in terms of the tangent values A, B, C and D. Two of these vertices have simple coordinates: $M(1, A)$ and $Q(1, -D)$. The other two are

$$N\left(\frac{1 - A^2 - 2AB}{1 + A^2}, \frac{2A + B - A^2B}{1 + A^2}\right) \quad \text{and} \quad P\left(\frac{1 - D^2 - 2CD}{1 + D^2}, \frac{C + 2D - CD^2}{1 + D^2}\right).$$

The crux of the proof lies in the following idea. Normally, we would look for four points (one on each side), which create the desired 2-by-2 grid partition. We would thus have four degrees of freedom (choosing the points) and four equations (the conditions that each of the smaller quadrilaterals formed is circumscribed).

However, the resulting algebraic system is extremely complicated. Trying to eliminate the unknowns one at a time leads to huge resultants which even MAPLE cannot handle.

Instead, we worked around this difficulty. Extend the sides of $MNPQ$ until they intersect at points E and F as shown in Figure 12 b). (Ignore the case when $MNPQ$ is a trapezoid for now). Now locate a point U on side MN and a point S on side QM such that when segments FU and ES are extended as in the figure, the four resulting quadrilaterals are all circumscribed. This reduces the number of variables from four to two and thus the system appears to be over-determined. However, extended investigations with Geometer's Sketchpad indicated that this construction is possible. At this point we start computing the coordinates of the newly introduced points. We have

$$E\left(\frac{1+AD}{1-AD}, \frac{A-D}{1-AD}\right) \quad \text{and} \quad F\left(1, \frac{A+B}{1-AB}\right).$$

Denote

$$m = \frac{MU}{MN} \qquad \text{and} \qquad q = \frac{QS}{QM}.$$

Clearly, the coordinates of U and V are rational functions on m, A, B, C and D while the coordinates of S and T depend in a similar manner on q, A, B, C and D. These expressions are quite complicated; for instance, each one of the coordinates of point T takes five full lines of MAPLE output. The situation is the same for the coordinates of point V.

Define the following quantities:

$$\begin{aligned} Z_1 &= MU + WS - WU - MS, \\ Z_2 &= NT + WU - WT - NU, \\ Z_3 &= PV + WT - WV - PT, \\ Z_4 &= QS + WV - WS - QV. \end{aligned}$$

By Theorem 2 b), a necessary and sufficient condition for the quadrilaterals $MUWS$, $NTWU$, $PVWT$ and $QSWV$ to be cyclic is that $Z_1 = Z_2 = Z_3 = Z_4 = 0$.

Notice that

$$Z_1 + Z_2 + Z_3 + Z_4 = MU - NU + NT - PT + PV - QV + QS - MS \quad (6)$$

and

$$\begin{aligned} Z_1 - Z_2 + Z_3 - Z_4 &= MN - NP + PQ - QM + 2(WS + WT - WU - WV) \\ &= 2(ST - UV), \end{aligned} \quad (7)$$

the last equality is due to the fact that $MNPQ$ is circumscribed.

Since we want $Z_i = 0$ for every $1 \leq i \leq 4$, we need to have the right hand terms from (6) and (7) each equal to 0. In other words, **necessary** conditions for finding the desired grid dissection are

$$MU - NU + PV - QV = PT - NT + MS - QS \qquad \text{and} \qquad UV = ST. \quad (8)$$

There is a two-fold advantage we gain by reducing the number of equations from four to two: first, the system is significantly simpler and second, we avoid using point W - the common vertex of all four small quadrilaterals which is also the point with the most complicated coordinates.

System (8) has two equations and two unknowns - m and q - and it is small enough for MAPLE to handle. Still, after eliminating variable q, the resultant is a polynomial of degree 10 of m with polynomial functions of A, B, C and D as coefficients.

This polynomial can be factored and the value of m we are interested in is a root of a quadratic. Although m does not have a rational expression depending on A, B, C and D it can still be written in terms of $\sqrt{\sin a}$, $\sqrt{\cos a}$, ..., $\sqrt{\sin d}$, $\sqrt{\cos d}$.

Explicit Formulation of Theorem 8. Let $MNPQ$ be a circumscribed quadrilateral as described in figure 12. Denote

$$s_1 = \sqrt{\sin a}, \quad s_2 = \sqrt{\sin b}, \quad s_3 = \sqrt{\sin c}, \quad s_4 = \sqrt{\sin d}$$
$$c_1 = \sqrt{\cos a}, \quad c_2 = \sqrt{\cos b}, \quad c_3 = \sqrt{\cos c}, \quad c_4 = \sqrt{\cos d}.$$

Define points $U \in MN$, $T \in NP$, $V \in TQ$ and $S \in QM$ such that

$$\frac{MU}{MN} = m, \quad \frac{NT}{NP} = n, \quad \frac{PV}{PQ} = p, \quad \frac{QS}{QM} = q$$

where

$$m = \frac{s_2 s_4 c_2^2 (s_4 s_1 (s_1^2 c_2^2 + s_2^2 c_1^2) + s_2 s_3)}{(s_1^2 c_2^2 + s_2^2 c_1^2)(s_1 s_2 s_4 + s_3 c_1^2)(s_1 s_2 s_3 + s_4 c_2^2)}, \tag{9}$$

$$n = \frac{s_3 s_1 c_3^2 (s_1 s_2 (s_2^2 c_3^2 + s_3^2 c_2^2) + s_3 s_4)}{(s_2^2 c_3^2 + s_3^2 c_2^2)(s_2 s_3 s_1 + s_4 c_2^2)(s_2 s_3 s_4 + s_1 c_3^2)}, \tag{10}$$

$$p = \frac{s_4 s_2 c_4^2 (s_2 s_3 (s_3^2 c_4^2 + s_4^2 c_3^2) + s_4 s_1)}{(s_3^2 c_4^2 + s_4^2 c_3^2)(s_3 s_4 s_2 + s_1 c_3^2)(s_3 s_4 s_1 + s_2 c_4^2)}, \tag{11}$$

$$q = \frac{s_1 s_3 c_1^2 (s_3 s_4 (s_4^2 c_1^2 + s_1^2 c_4^2) + s_1 s_2)}{(s_4^2 c_1^2 + s_1^2 c_4^2)(s_4 s_1 s_3 + s_2 c_4^2)(s_4 s_1 s_2 + s_3 c_1^2)}. \tag{12}$$

Denote $W = ST \cap UV$. Then, quadrilaterals $MUWS$, $NTWU$, $PVWT$ and $QSWV$ are all circumscribed (*i.e.*, $Z_1 = Z_2 = Z_3 = Z_4 = 0$).
Verifying these assertions was done in MAPLE. Recall that m and q were obtained as solutions of the system $Z_1 + Z_2 + Z_3 + Z_4 = 0$, $Z_1 - Z_2 + Z_3 - Z_4 = 0$. At this point it is not clear why for these choices of m, n, p and q we actually have $Z_i = 0$, for all $1 \le i \le 4$.

Using the expressions of m, n, p and q given above, we can write the coordinates of all points that appear in figure 12 in terms of s_i and c_i where $1 \le i \le 4$. We can then calculate the lengths of all the twelve segments which appear as sides of the smaller quadrilaterals.

For instance we obtain:

$$MU = \frac{(s_1^3 s_4 c_2^2 + s_1 s_2^2 s_4 c_1^2 + s_2 s_3) s_2 s_4}{c_1^2 (s_1 s_2 s_3 + c_2^2 s_4)(s_1 s_2 s_4 + c_1^2 s_3)},$$
$$NU = \frac{(s_2^3 s_3 c_1^2 + s_1^2 s_2 s_3 c_2^2 + s_1 s_4) s_1 s_3}{c_2^2 (s_1 s_2 s_3 + c_2^2 s_4)(s_1 s_2 s_4 + c_1^2 s_3)}.$$

and similar relations can be written for NT, PV, QS and PT, QV, MS by circular permutations of the expressions for MU and NU, respectively.

In the same way it can be verified that

$$UV = ST$$
$$= \frac{(s_1^2 c_4^2 + s_4^2 c_1^2)(s_1^2 s_4^2 + s_3^2 s_2^2 + 2 s_1 s_2 s_3 s_4 (s_1^2 c_2^2 + s_2^2 c_1^2))(s_3^2 c_1^2 + s_1^2 c_3^2 + 2 s_1 s_2 s_3 s_4)}{(s_1 s_2 s_3 + c_2^2 s_4)(s_2 s_3 s_4 + c_3^2 s_1)(s_3 s_4 s_1 + c_4^2 s_2)(s_1 s_2 s_4 + c_1^2 s_3)}$$

and

$$UW = \frac{\lambda_r \cdot UV}{\lambda_r + \mu_r}, \quad VW = \frac{\mu_r \cdot UV}{\lambda_r + \mu_r}, \quad SW = \frac{\lambda_s \cdot ST}{\lambda_s + \mu_s}, \quad TW = \frac{\mu_s \cdot ST}{\lambda_s + \mu_s},$$

where

$$\begin{aligned}
\lambda_r &= (s_1^3 s_2 c_4^2 + s_1 s_2 s_4^2 c_1^2 + s_3 s_4)(s_2 s_3 s_4 + s_1 c_3^2)(s_3 s_4 s_1 + s_2 c_4^2) \\
\mu_r &= (s_3^3 s_4 c_2^2 + s_3 s_4 s_2^2 c_3^2 + s_1 s_2)(s_4 s_1 s_2 + s_3 c_1^2)(s_1 s_2 s_3 + s_4 c_2^2) \\
\lambda_s &= (s_4^3 s_1 c_3^2 + s_4 s_1 s_3^2 c_4^2 + s_2 s_3)(s_2 s_3 s_4 + s_1 c_3^2)(s_1 s_2 s_3 + s_4 c_2^2) \\
\mu_s &= (s_2^3 s_3 c_1^2 + s_2 s_3 s_1^2 c_2^2 + s_4 s_1)(s_4 s_1 s_2 + s_3 c_1^2)(s_3 s_4 s_1 + s_2 c_4^2).
\end{aligned}$$

Still, verifying that $Z_i = 0$ is not as simple as it may seem. The reason is that the quantities s_i and c_i are not independent. For instance we have $s_i^4 + c_i^4 = 1$, for all $1 \le i \le 4$. Also, since $a+b+c+d = 180°$ we have $\sin(a+b) = \sin(c+d)$ which translates to $s_1^2 c_2^2 + s_2^2 c_1^2 = s_3^2 c_4^2 + s_4^2 c_3^2$. Similarly, $\cos(a+b) = -\cos(c+d)$ which means $c_1^2 c_2^2 - s_1^2 s_2^2 = s_3^2 s_4^2 - c_3^2 c_4^2$. There are $4+3+3 = 10$ such side relations which have to be used to prove that two expressions which look different are in fact equal. MAPLE cannot do this directly.

For example, it is not at all obvious that the expressions of m, n, p and q defined above represent numbers from the interval $(0, 1)$. Since each expression is obtained via circular permutations from the preceding one it is enough to look at m.

Clearly, since $s_i > 0$ and $c_i > 0$ for all $1 \le i \le 4$ we have that $m > 0$. On the other hand, using the side relations we mentioned above we get that

$$1 - m = \frac{s_3 s_1 c_1^2 (s_2 s_3 (s_1^2 c_2^2 + s_2^2 c_1^2) + s_1 s_4)}{(s_1^2 c_2^2 + s_2^2 c_1^2)(s_1 s_2 s_4 + s_3 c_1^2)(s_1 s_2 s_3 + s_4 c_2^2)}.$$

Obviously, $1 - m > 0$ and therefore $0 < m < 1$.

As previously eluded the construction works in the case when $MNPQ$ is a trapezoid as well. In this case if $MN \parallel PQ$ then $UV \parallel MN$ too. In conclusion, it is quite tricky to check that the values of m, n, p and q given by equalities (9) - (12) imply that $Z_1 = Z_2 = Z_3 = Z_4 = 0$. The MAPLE file containing the complete verification of theorem 8 is about 15 pages long. On request, we would be happy to provide a copy. □

4. Conclusions and Directions of Future Research

In this paper we mainly investigated what types of geometric properties can be preserved when dissecting a convex quadrilateral. The original contributions are contained in section 3 in which we dealt exclusively with grid dissections. There are many very interesting questions which are left unanswered.

1. The results from Theorems 6 and 7 suggest that if a cyclic quadrilateral $ABCD$ has an m-by-n grid dissection into cyclic quadrilaterals with $m \cdot n$ a large odd integer, then $ABCD$ has to be “close” to a rectangle. It would be desirable to quantify this relationship.

2. Conjecture 1 implies that orthodiagonal quadrilaterals are "bad" when it comes to class preserving dissections. On the other hand, theorem 8 proves that circumscribed quadrilaterals are very well behaved in this respect. Why does this happen? After all, the characterization Theorem 2 b) and 2 c) suggest that these two properties are not radically different.

More precisely, let us define an α-quadrilateral to be a convex quadrilateral $ABCD$ with $AB^\alpha + CD^\alpha = BC^\alpha + AD^\alpha$, where α is a real number. Notice that for $\alpha = 1$ we get the circumscribed quadrilaterals and for $\alpha = 2$ the orthodiagonal ones. In particular, a kite is an α-quadrilateral for all values of α. The natural question is:

Problem. For which values of α does every α-quadrilateral have a 2-by-2 grid dissection into α-quadrilaterals?

3. Theorem 8 provided a constructive method for finding a grid dissection of any circumscribed quadrilateral into smaller circumscribed quadrilaterals. Can this construction be extended to a 4-by-4 class preserving grid dissection? Notice that extending the opposite sides of each one of the four small cyclic quadrilaterals which appear in Figure 12 we obtain the same pair of points, E and F. It is therefore tempting to verify whether iterating the procedure used for $MNPQ$ for each of these smaller quads would lead to a 4-by-4 grid dissection of $MNPQ$ into 16 cyclic quadrilaterals. Maybe even a 2^n-by-2^n grid dissection is possible. If true, it is desirable to first find a simpler way of proving Theorem 8.

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