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A Simple Barycentric Coordinates Formula

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Abstract. We establish a simple formula for the barycentric coordinates with respect to a given triangle ABC of a point P specified by the oriented angles BPC, CPA and APB. Several applications are given.

We establish a simple formula for the homogeneous barycentric coordinates of a point with respect to a given triangle.

Theorem 1. *With reference to a given a triangle ABC, a point P specified by the oriented angles*

$$x = \measuredangle BPC, \quad y = \measuredangle CPA, \quad z = \measuredangle APB,$$

has homogeneous barycentric coordinates

$$\left(\frac{1}{\cot A - \cot x} : \frac{1}{\cot B - \cot y} : \frac{1}{\cot C - \cot z}\right). \tag{1}$$

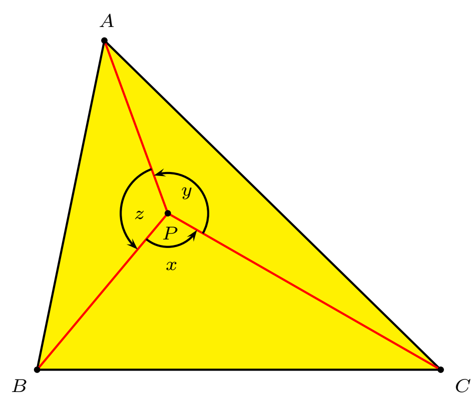

Figure 1.

Proof. Construct the circle through B, P, C, and let it intersect the line AP at A' (see Figure 2). Clearly, $\angle A'BC = \angle A'PC = \pi - \angle CPA = \pi - y$ and similarly, $\angle A'CB = \pi - z$. It follows from Conway's formula [5, §3.4.2] that in barycentric coordinates

$$A' = (-a^2 : S_C + S_{\pi-z} : S_B + S_{\pi-y}) = (-a^2 : S_C - S_z : S_B - S_y).$$

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Similarly, the lines BP intersects the circle CPA at a point B', and CP intersects the circle APB at C' whose coordinates can be easily written down. These be reorganized as

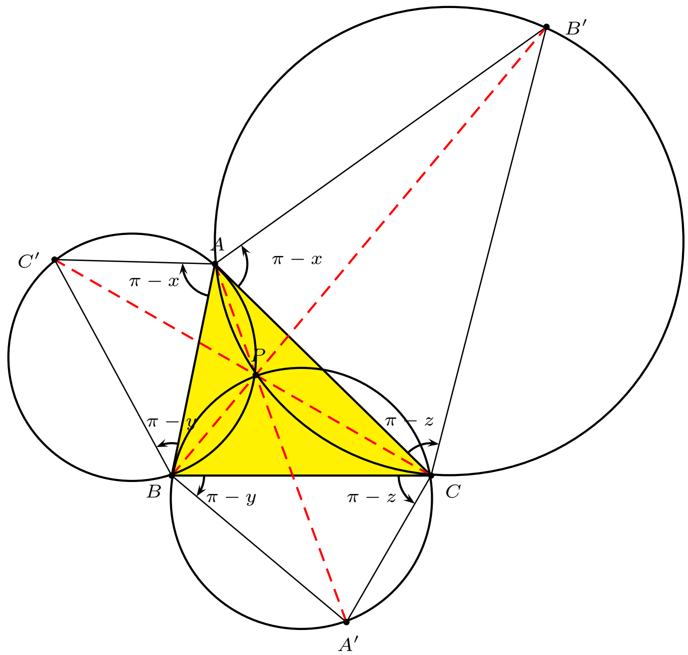

Figure 2.

$$A' = \left(-\frac{a^2}{(S_B - S_y)(S_C - S_z)} : \frac{1}{S_B - S_y} : \frac{1}{S_C - S_z}\right),$$
$$B' = \left(\frac{1}{S_A - S_x} : -\frac{b^2}{(S_C - S_z)(S_A - S_x)} : \frac{1}{S_C - S_z}\right),$$
$$C' = \left(\frac{1}{S_A - S_x} : \frac{1}{S_B - S_y} : -\frac{c^2}{(S_A - S_x)(S_B - S_y)}\right).$$

According the version of Ceva's theorem given in [5, §3.2.1], the lines AA', BB', CC' intersect at a point, which is clearly P, whose coordinates are

$$\left(\frac{1}{S_A - S_x} : \frac{1}{S_B - S_y} : \frac{1}{S_C - S_z}\right).$$

Since by definition $S_\theta = S \cdot \cot\theta$, this formula is clearly equivalent to (1). □

Remark. This note is a revision of [1]. Antreas Hatzipolakis has subsequently given a traditional trigonometric proof [3].

The usefulness of formula (1) is that it is invariant when we substitute x, y, z by directed angles.

Corollary 2 (Schaal)**.** *If for three points A', B', C' the directed angles $x = (A'B,\ A'C)$, $y = (B'C,\ B'A)$ and $z = (C'A,\ C'B)$ satisfy $x+y+z \equiv 0 \mod \pi$, then the circumcircles of triangles $A'BC$, $B'CA$, $C'AB$ are concurrent at P.*

Proof. Referring to Figure 2, if the circumcircles of triangles $A'BC$ and $B'CA$ intersect at P, then from concyclicity,

$$\begin{aligned}(PB,\ PC) = &\ (A'B,\ A'C) = x,\\ (PC,\ PA) = &\ (B'C,\ B'A) = y.\end{aligned}$$

It follows that

$$(PA,\ PB) = (PA,\ PC) + (PC,\ PB) = -y - x \equiv z = (C'A,\ C'B) \bmod \pi,$$

and C', A, B, P are concyclic. Now, it is obvious that the barycentrics of P are given by (1). □

For example, if the triangles $A'BC$, $B'CA$, $C'AB$ are equilateral on the exterior of triangle ABC, then $x = y = z = -\frac{\pi}{3}$, and $x + y + z \equiv 0 \bmod \pi$. By Corollary 2, we conclude that the circumcircles of these triangles are concurrent at

$$\begin{aligned}P = &\left(\frac{1}{\cot A - \cot\left(-\frac{\pi}{3}\right)} : \frac{1}{\cot B - \cot\left(-\frac{\pi}{3}\right)} : \frac{1}{\cot C - \cot\left(-\frac{\pi}{3}\right)}\right)\\ = &\left(\frac{1}{\cot A + \cot\left(\frac{\pi}{3}\right)} : \frac{1}{\cot B + \cot\left(\frac{\pi}{3}\right)} : \frac{1}{\cot C + \cot\left(-\frac{\pi}{3}\right)}\right).\end{aligned}$$

This is the first Fermat point, X_{13} of [4].

Corollary 3 (Hatzipolakis [2])**.** *Given a reference triangle ABC and two points P and Q, let R_a be the intersection of the reflections of the lines BP, CP in the lines BQ, CQ respectively* (see Figure 3). *Similarly define the points R_b and R_c. The circumcircles of triangles R_aBC, R_bCA, R_cAB are concurrent at a point*

$$f(P,Q) = \left(\frac{1}{\cot A - \cot(2x' - x)} : \frac{1}{\cot B - \cot(2y' - y)} : \frac{1}{\cot C - \cot(2z' - z)}\right), \tag{2}$$

where

$$\begin{aligned}x &= (PB,\ PC), \qquad y = (PC,\ PA), \qquad z = (PA,\ PB);\\ x' &= (QB,\ QC), \qquad y' = (QC,\ QA), \qquad z' = (QA,\ QB).\end{aligned} \tag{3}$$

Proof. Let $x'' = (R_aB,\ R_aC)$. Note that

$$\begin{aligned}x'' = &\ (R_aB,\ QB) + (QB,\ QC) + (QC,\ R_aC)\\ = &\ (QB,\ QC) + (R_aB,\ QB) + (QC,\ R_aC)\\ = &\ (QB,\ QC) + (QB,\ PB) + (PC,\ QC)\\ = &\ (QB,\ QC) + (QB,\ QC) - (PB,\ PC)\\ = &\ 2x' - x.\end{aligned}$$

Similarly, $y'' = (R_bC,\ R_bA) = 2y' - y$ and $z'' = (R_cA,\ R_cB) = 2z' - z$. Hence,

$$x'' + y'' + z'' \equiv 2(x' + y' + z') - (x + y + z) \equiv 0 \bmod \pi.$$

By Corollary 2, the circumcircles of triangles R_aBC, R_bCA, R_cAB are concurrent at the point $R = f(P,Q)$ given by (2). □

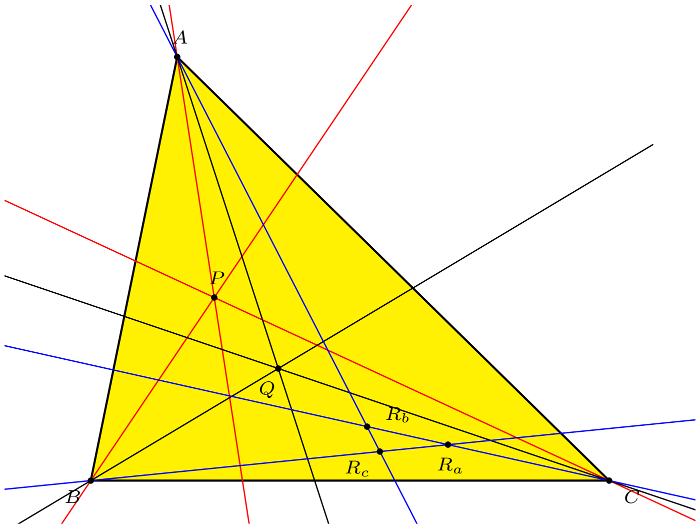

Figure 3.

Clearly, for the incenter I, $f(P, I) = P^*$, since $R_a = R_b = R_c = P^*$, the isogonal conjugate of P.

Corollary 4. *The mapping f preserves isogonal conjugation,* i.e.,

$$f^*(P, Q) = f(P^*, Q^*).$$

Proof. If the points P and Q are defined by the directed angles in (3), and $R = f(P, Q)$, $S = f(P^*,\ Q^*)$, then by Corollary 3, $(R^*B,\ R^*C) = A - (2x' - x)$ and

$$\begin{aligned}(SB,\ SC) &\equiv\ 2(Q^*B,\ Q^*C) - (P^*B,\ P^*C)\\ &\equiv\ 2(A - x') - (A - x)\\ &\equiv\ A - (2x' - x)\\ &\equiv\ (R^*B,\ R^*C) \bmod \pi.\end{aligned}$$

Similarly, $(SC, SA) \equiv (R^*C,\ R^*A)$ and $(SA,\ SB) \equiv (R^*A,\ R^*B) \bmod \pi$. Hence, $R^* = S$, or $f^*(P, Q) = f(P^*, Q^*)$. □

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