Forum Geometricorum
Volume 9 (2009) 275–278.

FORUM GEOM
ISSN 1534-1178

On the Possibility of Trigonometric Proofs of the Pythagorean Theorem

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Abstract. The identity $\cos^2 x + \sin^2 x = 1$ can be derived independently of the Pythagorean theorem, despite common beliefs to the contrary.

1. Introduction

In a remarkable 1940 treatise entitled *The Pythagorean Proposition*, Elisha Scott Loomis (1852–1940) presented literally hundreds of distinct proofs of the Pythagorean theorem. Loomis provided both "algebraic proofs" that make use of similar triangles, as well as "geometric proofs" that use area reasoning. Notably, none of the proofs in Loomis's book were of a style one would be tempted to call "trigonometric". Indeed, toward the end of his book ([1, p.244]) Loomis asserted that all such proofs are circular:

> There are no trigonometric proofs [of the Pythagorean theorem], because all of the fundamental formulae of trigonometry are themselves based upon the truth of the Pythagorean theorem; because of this theorem we say $\sin^2 A + \cos^2 A = 1$ etc.

Along the same lines but more recently, in the discussion page behind Wikipedia's Pythagorean theorem entry, one may read that a purported proof was once deleted from the entry because it "...depend[ed] on the veracity of the identity $\sin^2 x + \cos^2 x = 1$, which is the Pythagorean theorem ..." ([5]).

Another highly ranked Internet resource for the Pythagorean theorem is Cut-The-Knot.org, which lists dozens of interesting proofs ([2]). The site has a page devoted to fallacious proofs of the Pythagorean theorem. On this page it is again asserted that the identity $\cos^2 x + \sin^2 x = 1$ cannot be used to prove the Pythagorean theorem, because the identity "is based on the Pythagorean theorem, to start with" ([3]).

All of these quotations seem to reflect an implicit belief that the relation $\cos^2 x + \sin^2 x = 1$ cannot be derived independently of the Pythagorean theorem. For the record, this belief is false. We show in this article how to derive this identity independently of the Pythagorean theorem.

Publication Date: October 26, 2009. Communicating Editor: Paul Yiu.

The author would like to thank an anonymous referee for a number of suggestions, both general and specific, which greatly improved the manuscript during the editorial process.

2. Sine and cosine of acute angles

We begin by defining the sine and cosine functions for positive acute angles, independently of the Pythagorean theorem, as ratios of sides of similar right triangles. Given $\alpha \in \left(0,\ \frac{\pi}{2}\right)$, let $\mathcal{R}_\alpha$ be the set of all right triangles containing an angle of measure α, and let $\mathbf{T}$ be one such triangle. Because the angle measures in $\mathbf{T}$ add up to π (see Euclid's *Elements*, I.32), [1] $\mathbf{T}$ must have angle measures $\frac{\pi}{2}$, $\frac{\pi}{2} - \alpha$ and α. The side opposite to the right angle is the longest side (see *Elements*, I.19), called the hypotenuse of the right triangle; we denote its length by $H_{\mathbf{T}}$.

First consider the case $\alpha \neq \frac{\pi}{4}$. The three angle measures of $\mathbf{T}$ are distinct, so that the three side lengths are also distinct (see *Elements*, I.19). Let $A_{\mathbf{T}}$ denote the length of the side of $\mathbf{T}$ adjacent to the angle of measure α, and $O_{\mathbf{T}}$ the length of the opposite side. If $\mathbf{T}$ and $\mathbf{S}$ are any two triangles in $\mathcal{R}_\alpha$, then because $\mathbf{T}$ and $\mathbf{S}$ have angles of equal measures, corresponding side ratios in $\mathbf{S}$ and $\mathbf{T}$ are equal:

$$\frac{A_{\mathbf{T}}}{H_{\mathbf{T}}} = \frac{A_{\mathbf{S}}}{H_{\mathbf{S}}} \quad \text{and} \quad \frac{O_{\mathbf{T}}}{H_{\mathbf{T}}} = \frac{O_{\mathbf{S}}}{H_{\mathbf{S}}}$$

(see *Elements*, VI.4). Therefore, for $\alpha \neq \frac{\pi}{4}$ in the range $\left(0,\ \frac{\pi}{2}\right)$, we may define

$$\cos\alpha := \frac{A}{H} \quad \text{and} \quad \sin\alpha := \frac{O}{H},$$

where the ratios may be computed using any triangle in $\mathcal{R}_\alpha$. [2]

We next consider the case $\alpha = \frac{\pi}{4}$. Any right triangle containing an angle of measure $\frac{\pi}{4}$ must in fact have two angles of measure $\frac{\pi}{4}$ (see *Elements*, I.32), so its three angles have measures $\frac{\pi}{2}$, $\frac{\pi}{4}$ and $\frac{\pi}{4}$. Such a triangle is isosceles (see *Elements*, I.6), and therefore has only two distinct side lengths, H and L, where $H > L$ is the length of the side opposite to the right angle and L is the common length shared by the two other sides (see *Elements*, I.19). Because any two right triangles containing angle $\alpha = \frac{\pi}{4}$ have the same three angle measures, any two such triangles are similar and have the same ratio $\frac{L}{H}$ (see *Elements*, VI.4). Now therefore define

$$\cos\frac{\pi}{4} := \frac{L}{H} \quad \text{and} \quad \sin\frac{\pi}{4} := \frac{L}{H},$$

where again the ratios may be computed using any triangle in $\mathcal{R}_{\pi/4}$.

The ratios $\frac{A}{H}$, $\frac{O}{H}$, and $\frac{L}{H}$ are all strictly positive, for the simple reason that a triangle always has sides of positive length (at least in the simple conception of a triangle that operates here). These ratios are also all strictly less than unity, because H is the longest side (*Elements*, I.19 again). Altogether then, we have defined the

[1] The Pythagorean theorem is proved in Book I of the *Elements* as Proposition I.47, and the theorem is proved again in Book VI using similarity arguments as Proposition VI.31. References to the *Elements* should not be taken to mean that we are adopting a classical perspective on geometry. The references are only meant to reassure the reader that the annotated claims do not rely on the Pythagorean theorem (by showing that they precede the Pythagorean theorem in Euclid's exposition).

[2] We shall henceforth assume that for any $\alpha \in \left(0,\ \frac{\pi}{2}\right)$, there exists a right triangle containing an angle of measure α. The reader wishing to adopt a more cautious or classical viewpoint may replace the real interval $\left(0,\ \frac{\pi}{2}\right)$ everywhere throughout the paper by the set $\langle 0,\ \frac{\pi}{2}\rangle$ defined as the set of all $\alpha \in \left(0,\ \frac{\pi}{2}\right)$ for which there exists a right triangle containing an angle of measure α.

functions $\cos : \left(0,\ \frac{\pi}{2}\right) \to (0,1)$ and $\sin : \left(0,\ \frac{\pi}{2}\right) \to (0,1)$ independently of the Pythagorean theorem.

Because sine and cosine as defined above are independent of the Pythagorean theorem, any proof of the Pythagorean theorem may validly employ these functions. Indeed, *Elements*, VI.8 very quickly leads to the Pythagorean theorem with the benefit of trigonometric notation. [3] However, our precise concern in this paper is to derive trigonometric identities, and to this we now turn.

3. Subtraction formulas

The sine and cosine functions defined above obey the following subtraction formulas, valid for all $\alpha,\ \beta \in \left(0,\ \frac{\pi}{2}\right)$ with $\alpha - \beta$ also in $\left(0,\ \frac{\pi}{2}\right)$:

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta, \tag{1}$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{2}$$

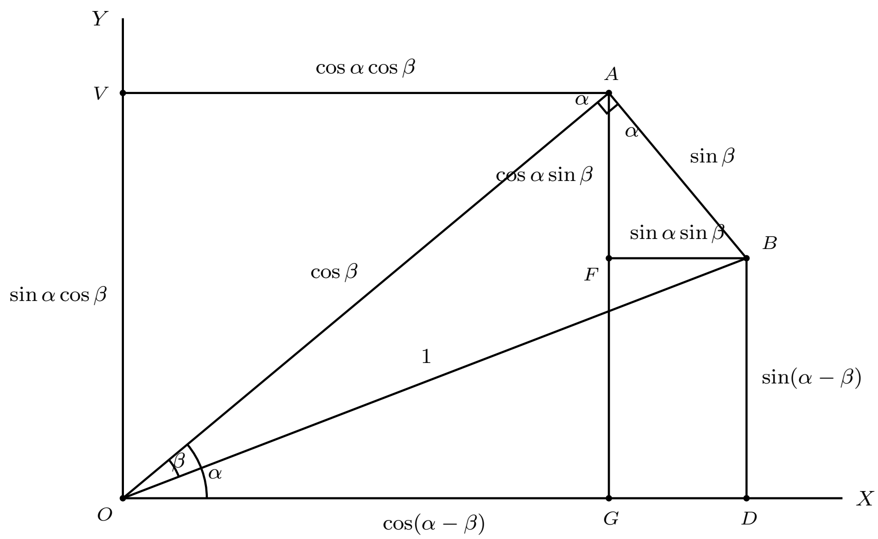

Figure 1.

The derivation of these formulas, as illustrated in Figure 1, is a textbook exercise. It is independent of the Pythagorean theorem, for although there are three hypotenuses OA, OB, and AB, their lengths are not calculated from the Pythagorean theorem, but rather from the sine and cosine we have just defined. Thus, assigning $OB = 1$, we have $OA = \cos\beta$ and $AB = \sin\beta$. The lengths of the horizontal and vertical segments are easily determined as indicated in Figure 1.

[3] See proof #6 in [2], specifically the observation attributed to R. M. Mentock.

4. The Pythagorean identity from the subtraction formula

It is tempting to try to derive the identity $\cos^2 x + \sin^2 x = 1$ by setting $\alpha = \beta = x$ and $\cos 0 = 1$ in (1). [4] This would not be valid, however, because the domain of the cosine function does not include zero. But there is a way around this problem. Given any $x \in \left(0,\ \frac{\pi}{2}\right)$, let y be any number with $0 < y < x < \frac{\pi}{2}$. [5] Then x, y, and $x - y$ are all in $\left(0,\ \frac{\pi}{2}\right)$. Therefore, applying (1) repeatedly, we have

$$\begin{aligned}
\cos y = &\ \cos(x - (x - y)) \\
= &\ \cos x \cos(x - y) + \sin x \sin(x - y) \\
= &\ \cos x(\cos x \cos y + \sin x \sin y) + \sin x(\sin x \cos y - \cos x \sin y) \\
= &\ (\cos^2 x + \sin^2 x) \cos y.
\end{aligned}$$

From this, $\cos^2 x + \sin^2 x = 1$.

5. Proving the Pythagorean theorem as a corollary

Because the foregoing proof is independent of the Pythagorean theorem, we may deduce the Pythagorean theorem as a corollary without risk of *petitio principii*. The identity $\cos^2 x + \sin^2 x = 1$ applied to a right triangle with legs a, b and hypotenuse c gives $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$, or $a^2 + b^2 = c^2$.

References

[1] E. S. Loomis, *The Pythagorean Proposition*, National Council of Teachers of Mathematics, 1968; available online at `http://eric.ed.gov/ERICDocs/data/ericdocs2sql/content_storage_01/0000019b/80/32/54/86.pdf`.

[2] `http://www.cut-the-knot.org/pythagoras/index.shtml`; last accessed 9/25/2009.

[3] `http://www.cut-the-knot.org/pythagoras/FalseProofs.shtml`; last accessed 9/25/2009.

[4] J. Versluys, *Zes en negentig bewijzen voor het Theorema van Pythagoras* (Ninety-Six Proofs of the Pythagorean Theorem), Amsterdam: A Versluys, 1914.

[5] `http://en.wikipedia.org/wiki/Talk:Pythagorean_theorem`; last accessed 9/25/2009.

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[4] A similar maneuver was attempted in 1914 by J. Versluys, who took $\alpha + \beta = \frac{\pi}{2}$ and $\sin \frac{\pi}{2} = 1$ in the addition formula for sine. Versluys cited Schur as the source of this idea (see [4, p.94]). A sign of trouble with the approach of Versluys/Schur is that the diagram typically used to derive the addition formula cannot be drawn for the case $\alpha + \beta = \frac{\pi}{2}$.

[5] If it is required that y belong to $\langle 0,\ \frac{\pi}{2} \rangle$ then one may put $y = \frac{x}{2}$.