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On the Construction of a Triangle from the Feet of Its Angle Bisectors

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Abstract. We give simple examples of triangles not constructible by ruler and compass from the feet of its angle bisectors when the latter form a triangle with an angle of 60° or 120°.

Given a triangle ABC with sides a, b, c, we want to construct a triangle $A'B'C'$ such that that segments AA', BB' and CC' are its angle bisectors, internal or external. Restricted to internal bisectors, this is Problem 138 of Wernick's list [3] (see also [2]). Yiu [4] has given a conic solution of the problem. Implicit in this is the impossibility of a ruler-and-compass construction in general, though in the case of a right angled triangle, this is indeed possible ([4, §7]). The purpose of this note is to give simple examples of $A'B'C'$ not constructible from ABC by ruler-and-compass when the latter contains a 60° or 120° angle.

Following [4] we denote by $(x : y : z)$ the barycentric coordinates of the incenter of triangle $A'B'C'$ with respect to triangle ABC, when A, B, C are the feet of the internal angle bisectors, or an excenter when one of A, B, C is the foot of an internal bisector and the remaining two external. The vertices of triangle $A'B'C'$ have coordinates $(-x, y, z)$, $(x, -y, z)$, $(x, y, -z)$. These coordinates satisfy the following equations (see [4, §3]):

$$\begin{aligned}
-x(c^2y^2 - b^2z^2) + yz((c^2 + a^2 - b^2)y - (a^2 + b^2 - c^2)z) &= 0,\\
-y(a^2z^2 - c^2x^2) + xz((a^2 + b^2 - c^2)z - (b^2 + c^2 - a^2)x) &= 0, \qquad (1)\\
-z(b^2x^2 - a^2y^2) + xy((b^2 + c^2 - a^2)x - (c^2 + a^2 - b^2)y) &= 0.
\end{aligned}$$

These three equations being dependent, it is enough to consider the last two. Elimination of z from these leads to a quartic equation in x and y. This fact already suggests the impossibility of a ruler-and-compass construction. However, this can be made precise if we put $c^2 = a^2 - ab + b^2$. In this case, angle C is 60° and we obtain, by writing $bx = t \cdot ay$, the following cubic equation in t:

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$$3(a-b)bt^3-(a^2-4ab+b^2)t^2+(a^2-4ab+b^2)t+3a(a-b)=0.$$

With $a=8$, $b=7$ (so that $c=\sqrt{57}$ and angle C is $60°$), this reduces to

$$7t^3+37t^2-37t+8=0,$$

which is easily seen not to have rational roots. The roots of the cubic equation are not constructible by ruler and compass (see [1, Chapter 3]). Explicit solutions can be realized by taking $A=(7,0)$, $B=(4,4\sqrt{3})$, $C=(0,0)$, with resulting $A'B'C'$ and the corresponding incenter (or excenter) exhibited in the table below.

t	$0.5492\cdots$	$0.3370\cdots$	$-6.1721\cdots$
A'	$(-0.3891, 6.8375)$	$(1.3112, 6.9711)$	$(5.8348, 0.7573)$
B'	$(1.4670, -25.7766)$	$(5.5301, 29.3999)$	$(7.6694, 0.9954)$
C'	$(8.5071, 7.0213)$	$(6.6557, 6.8857)$	$(6.3481, -0.9692)$
I	$(3.6999, 3.0537)$ incenter	$(3.7956, 3.9267)$ B' – excenter	$(3.7956, 3.9267)$ A' – excenter

On the other hand, if $c^2=a^2+ab+b^2$, the eliminant of z from (1) is also a cubic (in x and y) which, with the substitution $bx=t\cdot ay$, reduces to

$$3(a+b)bt^3-(a^2+4ab+b^2)t^2-(a^2-4ab+b^2)t+3a(a+b)=0.$$

With $a=2$, $b=1$ (so that $c=\sqrt{7}$ and angle C is $120°$), this reduces to

$$9t^3-13t^2-13t+18=0,$$

with three irrational roots. Explicit solutions can be realized by taking $A=(1,0)$, $B=(-1,\sqrt{3})$, $C=(0,0)$, with resulting $A'B'C'$ and the corresponding excenter exhibited in the table below.

t	$1.0943\cdots$	$1.5382\cdots$	$-1.1881\cdots$
A'	$(0.6876, -0.3735)$	$(5.2374, -2.2253)$	$(0.0436, 0.0549)$
B'	$(-1.4112, 0.7665)$	$(1.2080, -0.5132)$	$(-0.0555, -0.0699)$
C'	$(0.1791, 0.2609)$	$(0.7473, 0.6234)$	$(0.1143, -0.0586)$
I	$(0.1791, 0.2609)$ C' – excenter	$(0.3863, 0.3222)$ A' – excenter	$(-0.1261, 0.0646)$ C' – excenter

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